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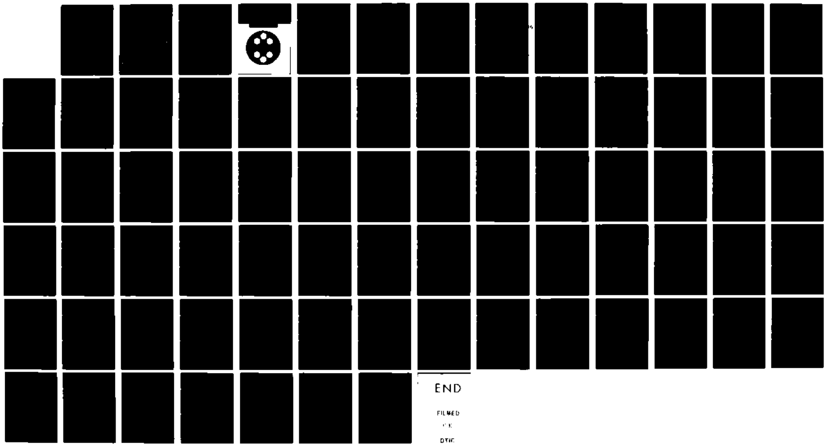

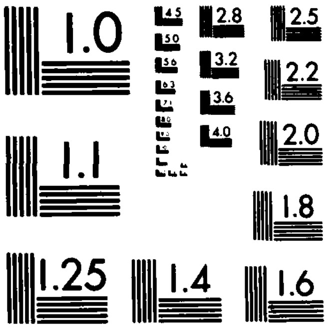

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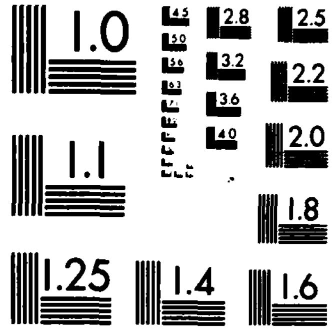

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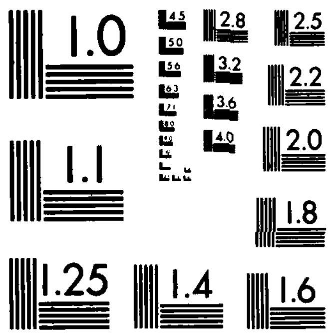

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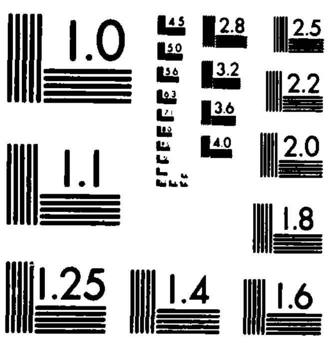

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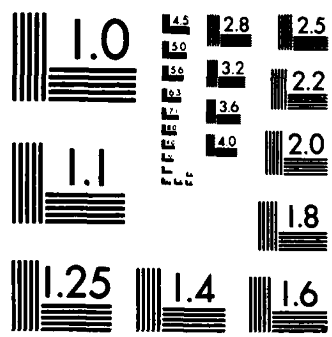

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ITEM #20, CONTINUED: sense, then the history value problem associated with our equation has a unique global smooth solution provided the initial history and forcing are suitably smooth and small. The proof combines a fixed point argument to establish local existence with a chain of global a priori 'energy-type' estimates.

The abstract results are then applied to establish global existence of smooth solutions to certain history value problems associated with the motion of nonlinear materials with fading memory, under assumptions which are realisitic within the framework of continuum mechanics.

Lefschetz Center for Dynamical Systems

A NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATION IN BANACH SPACE WITH APPLICATIONS TO MATERIALS WITH FADING MEMORY*

by

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August 1982

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A nonlinear functional differential equation in Banach Space with applications to materials with fading memory

by

William J. Hrusa

Abstract

We study a nonlinear functional differential equation in Banach space. This equation is an abstract form of the equations of motion for nonlinear materials with fading memory. Its basic structure is hyperbolic in character so that global smooth solutions should not be expected in general. Memory effects, however, may induce a dissipative mechanism which, although very subtle, is effective so long as the solution is small.

We show that if the memory is dissipative in an appropriate sense, then the history value problem associated with our equation has a unique global smooth solution provided the initial history and forcing are suitably smooth and small. The proof combines a fixed point argument to establish local existence with a chain of global a priori "energy-type" estimates.

The abstract results are then applied to establish global existence of smooth solutions to certain history value problems associated with the motion of nonlinear materials with fading memory, under assumptions which are realistic within the framework of continuum mechanics.

Chapter 1. Introduction

In continuum mechanics, the motion of a body is governed by a set of balance laws common to all continuous (mechanical) media, regardless of their composition. The type of material composing a body is characterized by a constitutive assumption which relates certain of the unknown fields appearing in the balance laws.

For nonlinear elastic materials, the balance laws lead to systems of quasilinear hyperbolic partial differential equations. A well-known feature of such systems is that they do not generally possess globally defined smooth solutions, no matter how smooth the initial data are. It seems interesting to consider situations where a constitutive assumption incorporates a dissipative mechanism in conjunction with an "elastic-type" response, and to study the effects of dissipation on solutions to the balance laws. In order to avoid purely technical complications and highlight the main ideas, we confine our study to motions which can be described with a single spatial coordinate.

Consider now the longitudinal motion of a one-dimensional body with reference configuration* $\mathcal{B}$, a connected open subset of $\mathbb{R}^1$. Let $u(x,t)$ denote the displacement at time t of the particle with reference position x (i.e. $x+u(x,t)$ is the position at time t of the particle with reference position x),

* We assume that the reference configuration is a natural state.

in which case the strain is given by* $\varepsilon(x,t) = u_x(x,t)$.

For simplicity, we assume that the body is homogeneous with unit reference density. The motion is then governed by

$$(1.1) \qquad u_{tt}(x,t) = \sigma_x(x,t) + f(x,t),$$

where σ is the stress and f is the (known) body force. A constitutive assumption relates the stress to the motion. We consider here only materials with the property that the stress at a material point x can be determined from the temporal history of the strain at x.

If the body is elastic, then $\sigma(x,t) = \phi(\varepsilon(x,t))$, where ϕ is an assigned smooth function with† $\dot{\phi}(0) > 0$, and the resulting equation of motion is

$$(1.2) \qquad u_{tt} = \phi(u_x)_x + f.$$

Lax [10] and MacCamy and Mizel [12] have shown that the initial value problem for (1.2) (with $f \equiv 0$) does not generally have a global (in time) smooth solution, no matter how smooth the initial data are.

For a viscoelastic body of the rate type, the stress at time t depends on the strain as well as the strain rate at time t. A typical constitutive assumption is $\sigma(x,t) = \phi(\varepsilon(x,t)) + \lambda\varepsilon_t(x,t)$, where ϕ is as before and λ is a

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* Here and throughout, subscripts x and t indicate partial derivatives.

† A dot is used to denote the derivative of a function of a single variable.

positive constant, which leads to the equation of motion

$$(1.3)\qquad u_{tt} = \phi(u_x)_x + \lambda u_{xtx} + f.$$

Global existence of smooth solutions to certain appropriate initial-boundary value problems for (1.3) has been established by Greenberg, MacCamy and Mizel [9], and Dafermos*[6], assuming ϕ, f, and the initial data are sufficiently smooth. Viscosity of the rate type is so powerful that global smooth solutions exist even if the initial data and body force are large.

A more subtle type of dissipation is induced by memory effects. For a material with memory, the stress at time t depends, in some fashion, on the history up to time t of the strain. If deformations that occurred in the distant past have less influence on the present stress than those which occurred in the recent past, we say the material has "fading memory".

A simple model for a material with fading memory is provided by linear viscoelasticity of the Boltzmann type, which is defined by the constitutive equation

$$(1.4)\qquad \sigma(x,t) = c\varepsilon(x,t) - \int_0^\infty g(s)\varepsilon(x,t-s)ds.$$

Here c is a positive constant and g is positive, decreasing, and

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* The results of Dafermos apply to the more general equation $u_{tt} = \Psi(u_x,u_{xt})_x + f$, with $\Psi_q(p,q) \geq k > 0$.

satisfies

$$(1.5) \qquad c - \int_0^\infty g(s)ds > 0.$$

Condition (1.5) has a natural mechanistic interpretation: In statics, i.e. $\varepsilon(x,t) = \bar{\varepsilon}(x)$ and $\sigma(x,t) = \bar{\sigma}(x)$ for all t, (1.4) reduces to

$$(1.6) \qquad \bar{\sigma}(x) = (c - \int_0^\infty g(s)ds)\bar{\varepsilon}(x).$$

Thus (1.5) states that the "equilibrium stress modulus" is positive.

In general, the constitutive equation for a material with memory takes the form

$$(1.7) \qquad \sigma(x,t) = \mathscr{G}(\varepsilon^t(x,\cdot)),$$

where for fixed x and t, $\varepsilon^t(x,\cdot)$ is the function mapping $[0,\infty)$ to $\mathbb{R}$ defined by $\varepsilon^t(x,s) = \varepsilon(x,t-s)$, $s \geq 0$, and $\mathscr{G}$ is a real-valued functional (not necessarily linear) with domain in an appropriate function space. The history of the strain up to some initial time is assumed to be known.

The notion of fading memory can be interpreted mathematically as a smoothness requirement for $\mathscr{G}$. Following Coleman and Noll [4,5], we introduce an influence function, intended to characterize the rate at which memory fades, and construct an L^p-type space of admissible strain histories, using the influence function as a weight. For convenience, we use history spaces of the L^2-type; our analysis can be adapted to L^p-type spaces for any p with $1 \le p < \infty$.

Let h be a positive, nonincreasing function belonging to $L^1(0,\infty)$, and denote by V_h the Banach space of all measurable functions $w:[0,\infty) \to \mathbb{R}$ such that $\int_0^\infty h(s)|w(s)|^2 ds < \infty$, equipped with the norm* given by

$$(1.8) \qquad \|w\|_h^2 = |w(0)|^2 + \int_0^\infty h(s)|w(s)|^2 ds.$$

We refer to h as an influence function and to the elements of V_h as histories. The reader is directed to Coleman and Mizel [3] for an axiomatic development of fading memory norms.

Formally, we say that a material has fading memory if the stress is determined by a constitutive equation of the form (1.7), and there exists an influence function h such that $\mathcal{G}$ is defined and continuously Fréchet differentiable on a neighborhood $\mathcal{O}$ of the zero history in V_h. This is essentially equivalent to the principle of fading memory formulated by Coleman and Noll [4,5].

The main focus of this investigation is on global existence of smooth solutions to the equations of motion for materials with fading memory. A typical problem of interest is to determine a smooth function u which satisfies†

---

* Functions in V_h are regarded as being equivalent if they are equal at 0 and equal almost everywhere on $(0,\infty)$.

† For $s \ge 0$, we set $u_x^t(x,s) = u_x(x,t-s)$.

$$(1.9) \qquad u_{tt}(x,t) = \frac{\partial}{\partial x} \mathscr{G}(u_x^t(x,\cdot)) + f(x,t),$$

$$x \in \mathscr{B},\ t > 0,$$

together with appropriate boundary conditions if $\mathscr{B}$ is bounded, and

$$(1.10) \qquad u(x,t) = v(x,t), \quad x \in \bar{\mathscr{B}},\ t \leq 0,$$

where v is an assigned smooth function. In order to h ~~l~~lobal existence, $\mathscr{G}$ must satisfy certain natural conditions.

Choose an influence function h and a neighborhood $\mathscr{O}$ of zero in V_h such that $\mathscr{G}$ is continuously differentiable on $\mathscr{O}$. It follows from the Riesz Representation Theorem that the Fréchet derivative of $\mathscr{G}$ admits representation

$$(1.11) \qquad \mathscr{G}'(w;\bar{w}) = E(w)\bar{w}(0) - \int_0^\infty K(w,s)\bar{w}(s)ds$$

for some $E:\mathscr{O} \to \mathbb{R}$ and $K:\mathscr{O}\times(0,\infty) \to \mathbb{R}$. We assume that K is continuous on $\mathscr{O}\times(0,\infty)$. Physically natural assumptions on E and K are that E(0) is positive and that $k(0,\cdot)$ is nonnegative, nonincreasing, and satisfies

$$(1.12) \qquad E(0) - \int_0^\infty K(0,s)ds > 0.$$

The interpretation of (1.12) is similar to that of (1.5). An elastic material is a special case of a material with fading memory, and consequently to have global existence, we must impose an additional restriction to ensure that the dependence of stress on past values of the strain is dissipative. To this end, we assume that $K(0,\cdot)$ does not vanish identically. Roughly speaking, the preceding conditions say that the linearization of (1.7) about the zero history is the constitutive relation for a physically reasonable linear viscoelastic material of the Boltzmann type. For technical reasons, we later strengthen the smoothness assumptions on F and K.

Coleman, Gurtin and Herrera [2], and Coleman and Gurtin [1] have studied wave propagation in materials with fading memory under essentially the above hypotheses. Of particular relevance to this work are the results of Coleman and Gurtin concerning the decay of acceleration waves, i.e. continuously differentiable solutions of (1.9) which sustain jump discontinuities in their second derivatives. The amplitude of an acceleration wave is defined to be the jump in acceleration. It is shown in [1] that if the amplitude of an acceleration wave is small initially, then it decays to zero monotonically. On the other hand, the amplitude of an acceleration wave may become infinite in finite time if the initial amplitude is large. This damping out of small discontinuities indicates the presence of a dissipative mechanism which is effective so long as the motion remains "small", and suggests that (1.9) should have global smooth solutions provided the initial history and forcing function are smooth and small.

Results of this type have been obtained by several authors for the model case

$$(1.13) \qquad \sigma(x,t) = \phi(\varepsilon^t(x,0)) - \int_0^\infty m(s)\psi(\varepsilon^t(x,s))ds$$

under appropriate conditions on ϕ, ψ, and m. For $\phi \equiv \psi$, existence theorems have been given by MacCamy [11], Dafermos and Nohel [7], and Staffans [15], and for ϕ different from ψ, by Dafermos and Nohel [8].

We here establish global existence and uniqueness of smooth solutions to a class of history value problems associated with (1.9) under smoothness and smallness assumptions on the data. In Chapter 2, we formulate a history value problem associated with an abstract version of (1.9). The assumptions used to analyze the abstract problem are motivated by mechanics. In Chapter 3, we prove the existence of a unique local solution defined on a maximal time interval. It is then shown, in Chapter 4, that the local solution is actually global if the initial history and forcing function are suitably small. In Chapter 5, the aforementioned abstract results are applied to the equations of motion for materials with fading memory.

The local existence argument is based on an application of the contraction mapping principle in an appropriate metric space. It does not rely on the history dependence being dissipative, and consequently applies to a larger class of materials. Global existence for small data is secured via a chain of a priori "energy-type" estimates. The presence of dissipation plays a crucial role in the development of these estimates. The basic strategy employed here of showing that dissipation prevails and ensures global existence when the data are small is due to Matsumura [13]. Finally, we remark that the pattern of estimates developed here was inspired by the paper of Dafermos and Nohel [8].

Chapter 2. Abstract Formulation

In this chapter, we formulate an abstract analogue of the history value problem (1.9),(1.10). We begin by discussing some preliminary notions.

A. Preliminaries

Let $\mathcal{Y}$ be a real Hilbert space with inner product $(\cdot,\cdot)$ and associated norm $\|\cdot\|$, and let I be an interval of real numbers. For m a nonnegative integer, we denote by $C^m(I;\mathcal{Y})$ the set of all functions mapping I to $\mathcal{Y}$ which, together with their first m derivatives (if $m \geq 1$), are bounded and continuous on the interior of I and admit continuous extensions to the closure of I. As usual, for $1 \leq p < \infty$, $L^p(I;\mathcal{Y})$ denotes the set of all (equivalence classes of) strongly measurable functions $w:I \to \mathcal{Y}$ such that $\int_I \|w(t)\|^p dt$ is finite, and $L^\infty(I;\mathcal{Y})$ is the set of all strongly measurable, essentially bounded functions mapping I to $\mathcal{Y}$. For m a nonnegative integer and $1 \leq p \leq \infty$, let $W^{m,p}(I;\mathcal{Y})$ be the Banach space of all functions $w:I \to \mathcal{Y}$ such that $\overset{(k)}{w} \in L^p(I;\mathcal{Y})$ for each k=0,1,...m, equipped with the norm defined by

$$\left(\sum_{k=0}^{m} \int_I \|\overset{(k)}{w}(t)\|^p dt\right)^{1/p} \qquad \text{if} \quad p < \infty,$$

and

$$\operatorname*{ess-sup}_{t\in I} \sum_{k=0}^{m} \|\overset{(k)}{w}(t)\| \qquad \text{if} \quad p = \infty.$$

Here $\overset{(k)}{w}$ denotes the k^{th} derivative* of w with the convention $\overset{(0)}{w} = w$. We often write $\dot{w}$ and $\ddot{w}$ in place of $\overset{(1)}{w}$ and $\overset{(2)}{w}$.

Let u be a function mapping some interval $(-\infty,T]$ into $\mathscr{Y}$. For each $t \in (-\infty,T]$ we define the function $u^t:[0,\infty) \to \mathscr{Y}$ by

$$(2.1) \qquad u^t(s) = u(t-s), \quad s \geq 0.$$

If $u:(-\infty,T] \to \mathscr{Y}$ is sufficiently smooth, then for each $t \in (-\infty,T]$, we define the functions $\dot{u}^t, \ddot{u}^t:[0,\infty) \to \mathscr{Y}$ by

$$(2.2) \qquad \dot{u}^t(s) = \dot{u}(t-s), \quad s \geq 0,$$

and

$$(2.3) \qquad \ddot{u}^t(s) = \ddot{u}(t-s), \quad s \geq 0,$$

etc.

Certain of the estimates in Chapters 3 and 4 can be derived by a formal computation, which, to be made rigorous, would require smoothness properties beyond those possessed by the functions involved. In these situations, we must first work with a discrete analogue of the estimate and then take limits. For this purpose we introduce the forward difference operator Δ_η of stepsize η. If $w:I \to \mathscr{Y}$, then for each $\eta > 0$, we define $\Delta_\eta w$ by

$$(2.4) \qquad (\Delta_\eta w)(t) = w(t+\eta)-w(t).$$

---

*These derivatives are to be understood in the sense of vector-valued distributions.

We shall make frequent use of many standard Hilbert space inequalities, particularly the Cauchy-Schwarz inequality and two of its immediate consequences,

$$(2.5)\qquad |(x,y)| \leq \frac{\varepsilon}{2}\|x\|^2 + \frac{1}{2\varepsilon}\|y\|^2 \qquad \forall\ x,y \in \mathscr{Y},\quad \varepsilon > 0,$$

and the so-called Cauchy inequality

$$(2.6)\qquad \Big\|\sum_{i=1}^{m} x_i\Big\|^2 \leq m \sum_{i=1}^{m}\|x_i\|^2, \qquad x_1,x_2,\ldots x_m \in \mathscr{Y}.$$

B. Basic Spaces

We now introduce certain spaces that will play a central role in the formulation and analysis of the abstract history value problem. Let $\mathscr{X}_k$, k=0,1,2,3, be real Hilbert spaces such that $\mathscr{X}_{k+1}$ is continuously and densely imbedded in $\mathscr{X}_k$ for k=0,1,2. We denote the norms and inner products on $\mathscr{X}_k$ by $\|\cdot\|_k$ and $\langle\cdot,\cdot\rangle_k$, respectively, for k=0,1,2,3. By $\mathscr{X}_{-1}$, we denote the dual of $\mathscr{X}_1$ constructed via the inner product on $\mathscr{X}_0$, i.e. $\mathscr{X}_{-1}$ is the completion of $\mathscr{X}_0$ under the norm defined by

$$(2.7)\qquad \|x\|_{-1} = \sup_{\|y\|_1=1} \langle x,y\rangle_0.$$

Clearly, $\mathscr{X}_0$ is imbedded continuously and densely in $\mathscr{X}_{-1}$. We assume that

$$(2.8)\qquad \langle x,y\rangle_2 \leq \|x\|_1\cdot\|y\|_3 \qquad \forall\ x,y \in \mathscr{X}_3,$$

which implies that every continuous bilinear form on $\mathscr{X}_2\times\mathscr{X}_2$ admits continuous extension to $\mathscr{X}_1\times\mathscr{X}_3$.

The only inner product which will be used explicitly in the sequel is $\langle\cdot,\cdot\rangle_0$. In order to simplify the notation, we drop the subscript and write $\langle\cdot,\cdot\rangle$ in place of $\langle\cdot,\cdot\rangle_0$. The symbol $\langle\cdot,\cdot\rangle$ will also be used to denote the duality pairing between $\mathscr{X}_{-1}$ and $\mathscr{X}_1$.

We shall frequently be concerned with functions which take values in this scale of spaces. If f maps an interval I into $\mathscr{X}_3$, it can also be regarded in a natural way as a mapping from I to $\mathscr{X}_k$ for k=-1,0,1,2. We use the same symbol to denote each of these maps. Similar comments apply to linear operators. To simplify our notation, we set*

$$\mathscr{L} = \bigcap_{k=1}^{3} \mathscr{L}(\mathscr{X}_k;\mathscr{X}_{k-2}), \tag{2.9}$$

equipped with the operator norm defined by

$$\|L\|_{\mathscr{L}} = \sup_{\|x\|_3=1} \|Lx\|_1 + \sup_{\|x\|_2=1} \|Lx\|_0 + \sup_{\|x\|_1=1} \|Lx\|_{-1}. \tag{2.10}$$

In the applications, the spaces $\mathscr{X}_k$ will be subspaces of the usual Sobolev spaces $W^{k,2}(\mathscr{B})$.

Let h be a fixed real-valued influence function, i.e. a nonincreasing real-valued function belonging to $L^1(0,\infty)$ with $h(s) > 0$ for all $s > 0$. Without loss of generality, we assume

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*As usual, $\mathscr{L}(\mathscr{X}_k;\mathscr{X}_{k-2})$ is the set of all bounded linear maps from $\mathscr{X}_k$ to $\mathscr{X}_{k-2}$.

$$(2.11) \qquad \int_0^\infty h(s)ds = 1.$$

For k=1,2,3, we denote by $\mathscr{V}_k$ the Hilbert space of all strongly measurable functions $w:[0,\infty) \to \mathscr{X}_k$ such that $\int_0^\infty h(s)\|w(s)\|_k^2 ds$ is finite, equipped with the norm* given by

$$(2.12) \qquad |\!|\!| w |\!|\!|_k^2 = \|w(0)\|_k^2 + \int_0^\infty h(s)\|w(s)\|_k^2 \, ds.$$

Clearly, $\mathscr{V}_{k+1}$ is imbedded continuously and densely in $\mathscr{V}_k$ for k=1,2. Moreover, (2.8) implies that continuous bilinear forms on $\mathscr{V}_2 \times \mathscr{V}_2$ and on $\mathscr{V}_2 \times \mathscr{X}_2$ admit continuous extensions to $\mathscr{V}_1 \times \mathscr{V}_3$ and $\mathscr{V}_1 \times \mathscr{X}_3$, respectively. When no confusion is likely to arise, we use the same symbol to denote a bilinear form and its extension.

It follows from the Lebesgue Dominated Convergence Theorem that if $w \in C^0((-\infty,T];\mathscr{X}_k)$ for some k=1,2,3, then the map $t \to w^t$ is a continuous mapping of $(-\infty,T]$ into $\mathscr{V}_k$. Also, if $w \in C^1((-\infty,T];\mathscr{X}_k)$ then the map $t \to w^t$ is continuously differentiable from $(-\infty,T]$ into $\mathscr{V}_k$ and

$$(2.13) \qquad \frac{d}{dt}(w^t) = \dot{w}^t , \qquad -\infty < t \le T.$$

If $w \in C^0((-\infty,T];\mathscr{X}_k)$ for some $T > 0$ and some k=1,2,3, one easily deduces that for each $t \in [0,T]$

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* We regard functions in $\mathscr{V}_k$ as being equivalent if they are equal at 0 and equal a.e. on $(0,\infty)$. This norm is associated in an obvious way with an inner product.

$$(2.14)\qquad |||w^t|||_k^2 \le 2 \sup_{s\in[0,t]} ||w(s)||_k^2 + |||w^0|||_k^2.$$

This inequality will prove useful in analyzing the history value problem which we are about to describe.

C. The History Value Problem

Let A be a smooth map from $\mathscr{V}_2$ to $\mathscr{L}$ and B be a smooth map from $\mathscr{V}_2\times[0,\infty)$ to $\mathscr{L}$. We seek a function u mapping $(-\infty,\infty)$ to the spaces $\mathscr{X}_k$ which satisfies

$$(2.15)\qquad \ddot{u}(t)+A(u^t)u(t) + \int_0^\infty B(u^t,s)u^t(s)ds = f(t), \quad t > 0,$$

and

$$(2.16)\qquad u(t) = v(t), \qquad -\infty < t \le 0.$$

Here f is a given function mapping $[0,\infty)$ to the spaces $\mathscr{X}_k$, and v is an assigned function on $(-\infty,0]$ with $v^0 \in \mathscr{V}_2$.

In the next section, we collect together all of the assumptions which we shall impose on A and B. However, before these assumptions are stated, the following remarks are in order.

It is not our intention here to develop a general theory for functional differential equations in Hilbert space; (2.15) should be regarded as an abstract version of (1.9). By studying (2.15) rather than analyzing (1.9) directly, we can develop an existence theorem for (2.15),(2.16) which will be applicable to a large class of problems associated with (1.9) and avoid the repetition of standard arguments. Also, a proof in the abstract framework offers certain notational conveniences, and, hopefully, is more illuminating.

We have tried to state our assumptions in such a way that they will be convenient for the proofs in Chapters 3 and 4 and will follow from a minimal set of assumptions in the applications; we have not tried to state a minimal set of assumptions on A and B. (See Remarks 2.1 and 2.2.). Although (a-1) through (a-11) may appear somewhat complicated, they will be satisfied in the applications under a rather simple set of conditions on $\mathcal{G}$, all of which are quite reasonable from the point of view of mechanics. Finally, we remark that even though we are imposing certain global conditions on A and B, our results are applicable to situations in mechanics where $\mathcal{G}$ is defined only on a neighborhood of zero in V_h. (See Theorem 5.1 et. seq.)

D. Basic Assumptions

Let the spaces $\mathcal{X}_k$, $\mathcal{L}$, and $\mathcal{V}_k$ be as described in Section B. We assume that:

(a-1): The map $A: \mathcal{V}_2 \to \mathcal{L}$ is twice continuously Fréchet differentiable* and there is a constant Λ_1 such that

$$(2.17)\quad \|A(w)\|_{\mathcal{L}} \le \Lambda_1 \qquad \forall\, w \in \mathcal{V}_2,$$

$$(2.18)\quad \|A'(w;z)\|_{\mathcal{L}} \le \Lambda_1 |||z|||_2 \qquad \forall\, w,z \in \mathcal{V}_2,$$

$$(2.19)\quad \|A''(w;z_1,z_2)\|_{\mathcal{L}} \le \Lambda_1 |||z_1|||_2 \cdot |||z_2|||_2 \qquad \forall\, w,z_1,z_2 \in \mathcal{V}_2.$$

* We use the notation A'(w;z) to denote the Fréchet derivative of A at w acting on z.

(a-2): There is a positive constant λ_1 such that

(2.20) $\langle A(w)x,x\rangle \geq \lambda_1 \|x\|_1^2 \qquad \forall\ w \in \mathscr{V}_2,\ x \in \mathscr{X}_1.$

(a-3): For each fixed $w \in \mathscr{V}_2$, the linear operator $A(w)$ is invertible with $(A(w))^{-1} \in \bigcap_{k=2}^{3} \mathscr{L}(\mathscr{X}_{k-2};\mathscr{X}_k)$ and there is a constant μ_1 such that

(2.21) $\|(A(w))^{-1}x\|_k \leq \mu_1 \|x\|_{k-2}, \quad k=2,3,\ \forall\ w \in \mathscr{V}_2, \quad x \in \mathscr{X}_1.$

(a-4): We define

(2.22) $a(w;x_1,x_2) = \langle A(w)x_1,x_2\rangle - \langle A(w)x_2,x_1\rangle$

and assume that there is a constant μ_2 such that

(2.23) $|a(w;x_1,x_2)| \leq \mu_2 \|x_1\|_0 \cdot \|x_2\|_1 \cdot (1+|||w|||_3^2)$

$\forall\ w \in \mathscr{V}_3,\ x_1,x_2 \in \mathscr{X}_1.$

(a-5): For each $w \in \bigcap_{k=0}^{3} W^{3-k,\infty}((-\infty,T];\mathscr{X}_k)$, $T > 0$,

$Z_0 \in \mathscr{X}_3$, $Z_1 \in \mathscr{X}_2$, and $g:[0,T] \to \mathscr{X}_0$ with

(2.24) $g \in C^0([0,T];\mathscr{X}_1) \cap C^1([0,T];\mathscr{X}_0),$

(2.25) $\ddot{g} \in L^2([0,T];\mathscr{X}_0),$

the linear initial value problem

(2.26) $\ddot{Z}(t)+A(w^t)Z(t) = g(t), \quad 0 \leq t \leq T,$

(2.27) $Z(0) = Z_0,\ \dot{Z}(0) = Z_1$

has a unique solution $Z \in \bigcap_{k=0}^{3} C^{3-k}([0,T];\mathscr{X}_k)$.

(a-6): The map $B: \mathscr{V}_2 \times [0,\infty) \to \mathscr{L}$ is (jointly) twice continuously Fréchet differentiable* and there is a constant Λ_2 such that

(2.28) $$\int_0^\infty \|B(w,s)\|_{\mathscr{L}}^2 h(s)^{-1} ds \le \Lambda_2 \qquad \forall\ w \in \mathscr{V}_2,$$

(2.29) $$\int_0^\infty \|\dot{B}(w,s)\|_{\mathscr{L}}^2 h(s)^{-1} ds \le \Lambda_2 \qquad \forall\ w \in \mathscr{V}_2,$$

(2.30) $$\int_0^\infty \|B'(w;z,s)\|_{\mathscr{L}}^2 h(s)^{-1} ds \le \Lambda_2 |||z|||_2^2 \qquad \forall\ w,z \in \mathscr{V}_2,$$

(2.31) $$\int_0^\infty \|\dot{B}'(w,z,s)\|_{\mathscr{L}}^2 h(s)^{-1} ds \le \Lambda_2 |||z|||_2^2 \qquad \forall\ w,z \in \mathscr{V}_2,$$

(2.32) $$\int_0^\infty \|B''(w;z_1,z_2,s)\|_{\mathscr{L}}^2 h(s)^{-1} ds \le \Lambda_2 |||z_1|||_2^2 \cdot |||z_2|||_2^2$$

$$\forall\ w,z_1,z_2 \in \mathscr{V}_2.$$

We define

(2.33) $$C(0,s) = \int_s^\infty B(0,\xi)d\xi, \qquad s \ge 0,$$

and

(2.34) $$F(0) = A(0) - C(0,0),$$

and assume that

(a-7): $$\int_0^\infty \|C(0,s)\|_{\mathscr{L}}^2 h(s)^{-1} ds < \infty .$$

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* Here $B'(\cdot;\cdot,s)$ denotes the Fréchet derivative of $B(\cdot,s)$ for fixed s, and $\dot{B}(w,s)$ is the derivative of $B(w,s)$ with respect to s for fixed w. We use $\dot{B}'$ to denote the "mixed" derivative.

(a-8): There is a positive constant λ_2 such that

(2.35) $$\langle F(0)x,x \rangle \geq \lambda_2 \|x\|_1^2 \qquad \forall\ x \in \mathscr{X}_1,$$

(2.36) $$\langle C(0,0)x,x \rangle \geq \lambda_2 \|x\|_1^2 \qquad \forall\ x \in \mathscr{X}_1,$$

(2.37) $$\langle C(0,0)x,F(0)x \rangle \geq \lambda_2 \|x\|_2^2 \quad \forall\ x \in \mathscr{X}_2,$$

(2.38) $$\|C(0,0)x\|_0 \geq \lambda_2 \|x\|_2 \qquad \forall\ x \in \mathscr{X}_2,$$

(2.39) $$\|F(0)x\|_0 \geq \lambda_2 \|x\|_2 \qquad \forall\ x \in \mathscr{X}_2.$$

(a-9): $C(0,0)$ and $F(0)$ satisfy

(2.40) $$\langle C(0,0)x,y \rangle = \langle C(0,0)y,x \rangle \qquad \forall\ x,y \in \mathscr{X}_2,$$

(2.41) $$\langle F(0)x,y \rangle = \langle F(0)y,x \rangle \qquad \forall\ x,y \in \mathscr{X}_2,$$

(2.42) $$\langle C(0,0)x,F(0)y \rangle = \langle C(0,0)y,F(0)x \rangle \qquad \forall\ x,y \in \mathscr{X}_2.$$

(a-10): There is a positive constant β such that

(2.43) $$\int_0^T \Big\| \int_0^t C(0,t-\xi)w(\xi)d\xi \Big\|_0^2\, dt \leq \beta \int_0^T \Big\langle C(0,0)w(t), \int_0^t C(0,t-\xi)w(\xi)d\xi \Big\rangle\, dt,$$

(2.44) $$\int_0^T \Big\| \int_0^t B(0,t-\xi)w(\xi) \Big\|_0^2 dt \leq \beta \int_0^T \Big\langle C(0,0)w(t), \int_0^t C(0,t-\xi)w(\xi)d\xi \Big\rangle\, dt$$

for every $w \in C^0([0,T];\mathscr{X}_2)$ and every $T > 0$.

(a-11): There is a constant μ_3 such that for every $w \in C^0([0,T];\mathscr{X}_1)$ and every $T > 0$, the linear integral equation

$$A(0)w(t) + \int_0^t B(0,t-\xi)w(\xi)d\xi = g(t) \tag{2.45}$$

has a unique solution $w \in C^0([0,T];\mathscr{X}_3)$ which satisfies

$$\sup_{t\in[0,T]} \|w(t)\|_3 \le \mu_3 \sup_{t\in[0,T]} \|g(t)\|_1, \tag{2.46}$$

$$\int_0^T \|w(t)\|_3^2\, dt \le \mu_3 \int_0^T \|g(t)\|_1^2. \tag{2.47}$$

Remark 2.1: Existence of solutions to the linear initial value problem (2.26),(2.27) can be proven in the abstract setting. However, the proof is rather lengthy and in the applications standard existence theory for linear hyperbolic equations will imply that (a-5) is satisfied.

Remark 2.2: Existence of solutions to the linear integral equation (2.45) can also be established in the abstract setting. However, in the applications one simple condition will guarantee that both (a-10) and (a-11) are satisfied.

Chapter 3. Local Existence

The objective of this chapter is to establish the existence of a unique local solution to the history value problem (2.15), (2.16). We assume throughout that the basic assumptions (a-1) through (a-6) hold. In addition, we assume that f satisfies

$$f \in C^0([0,\infty);\mathscr{X}_1) \cap C^1([0,\infty);\mathscr{X}_0), \tag{3.1}$$

$$\ddot{f} \in L^2([0,\infty);\mathscr{X}_0), \tag{3.2}$$

and that v satisfies

$$v \in \bigcap_{k=0}^{3} C^{3-k}((-\infty,0];\mathscr{X}_k), \tag{3.3}$$

and the compatibility conditions

$$\ddot{v}(0) = -A(v^0)v(0) - \int_0^\infty B(v^0,s)v^0(s)ds + f(0), \tag{3.4}$$

$$\begin{aligned} \overset{(3)}{v}(0) = -A(v^0)\dot{v}(0) &- A'(v^0;\dot{v}^0)v(0) \\ &- \int_0^\infty B(v^0,s)\dot{v}^0(s)ds \\ &- \int_0^\infty B'(v^0;\dot{v}^0,s)v^0(s)ds \\ &+ \dot{f}(0). \end{aligned} \tag{3.5}$$

The purpose of (3.4),(3.5) is to ensure that the solution will be smooth across t=0. An existence theory for (2.15), (2.16) can be developed without assuming (3.4),(3.5), however,

$\ddot{u}$ will generally be discontinuous at zero.

Theorem 3.1: Assume that the basic assumptions (a-1) through (a-6) hold, that f satisfies (3.1) and (3.2), and that v satisfies (3.3),(3.4), and (3.5). Then, the history value problem (2.15),(2.16) has a unique local solution u defined on a maximal interval $(-\infty,T_{max})$, $T_{max} > 0$, such that for each $T < T_{max}$, the restriction of u to $(-\infty,T]$ satisfies

$$(3.6) \qquad u \in \bigcap_{k=0}^{3} C^{3-k}((-\infty,T],\mathscr{X}_k).$$

Moreover, if $T_{max} < \infty$, then

$$(3.7) \qquad \sup_{s\in[0,t]} \sum_{k=0}^{3} \| \overset{(k)}{u}(s)\|^2_{3-k} \to \infty \quad \text{as} \quad t \uparrow T_{max}.$$

The proof of this theorem is rather lengthy and will be partitioned into several lemmas. We begin by constructing a metric space which will play a central role in the remainder of the chapter.

For M,T > 0, let $\mathscr{Y}(M,T)$ denote the set of all functions $w:(-\infty,T] \to \mathscr{X}_0$ which satisfy

$$(3.8) \qquad w \in \bigcap_{k=0}^{3} W^{3-k,\infty}((-\infty,T];\mathscr{X}_k),$$

$$(3.9) \qquad w(t) = v(t), \qquad t \leq 0,$$

and

$$(3.10) \qquad \operatorname*{ess\text{-}sup}_{t\in[0,T]} \sum_{k=0}^{3} \| \overset{(k)}{w}(t)\|^2_{3-k} \leq M^2.$$

Observe that $\mathscr{Y}(M,T)$ is nonempty if M is sufficiently large. Henceforth, we tacitly make this assumption.

Define a metric ρ on $\mathscr{Y}(M,T)$ by

$$(3.11) \qquad \rho(w_1,w_2) = \sup_{t\in[0,T]} \sum_{k=0}^{2} \| \overset{(k)}{w_1}(t) - \overset{(k)}{w_2}(t) \|_{2-k}^2 .$$

<u>Lemma 3.1</u>: Equipped with metric ρ, $\mathscr{Y}(M,T)$ becomes a complete metric space.

<u>Proof</u>: That ρ defines a metric on $\mathscr{Y}(M,T)$ is obvious. Suppose that $\{w_j\}_{j=1}^{\infty}$ is a Cauchy sequence in $(\mathscr{Y}(M,T),\rho)$. It then follows easily that there is a function w belonging to $\bigcap_{k=0}^{2} W^{2-k,\infty}((-\infty,T];\mathscr{X}_k)$, with $w(t) = v(t)$ for $t \leq 0$, such that $w_j \to w$ (strongly) in $W^{2-k,\infty}((-\infty,T];\mathscr{X}_k)$ for each k=0,1,2. On account of (3.10), there exists a subsequence $\{w_{j\ell}\}$ and a function x belonging to $\bigcap_{k=0}^{3} W^{3-k,\infty}((-\infty,T];\mathscr{X}_k)$, with $x(t) = v(t)$ for $t \leq 0$, such that $w_{j\ell} \rightharpoonup x$ weak * in $W^{3-k,\infty}((-\infty,T];\mathscr{X}_k)$ for k=0,1,2,3. From the sequential weak * lower semicontinuity property of norms, we deduce that

$$(3.12) \qquad \operatorname*{ess-sup}_{t\in[0,T]} \sum_{k=0}^{3} \| \overset{(k)}{x}(t) \|_{3-k}^2 \leq M^2,$$

whence $x \in \mathscr{Y}(M,T)$. By uniqueness of limits, we have $w = x$, and consequently $w \in \mathscr{Y}(M,T)$. (Indeed, weak * convergence in $W^{3,\infty}((-\infty,T];\mathscr{X}_0)$ and strong convergence in $W^{2,\infty}((-\infty,T];\mathscr{X}_0)$ both imply weak * convergence in $W^{2,\infty}((-\infty,T];\mathscr{X}_0)$, for example.)

Therefore, $w_j \to w$ in the space $(\mathscr{G}(M,T),\rho)$, which proves the lemma. ■

Now, for w in $\mathscr{G}(M,T)$, consider the initial value problem

$$(3.13)\qquad \ddot{Z}(t)+A(w^t)Z(t) + \int_0^\infty B(w^t,s)w^t(s)ds = f(t), \qquad 0 \le t \le T,$$

$$(3.14)\qquad Z(0) = v(0), \quad \dot{Z}(0) = \dot{v}(0).$$

If we set $g(t) = f(t) - \int_0^\infty B(w^t,s)w^t(s)ds$, $0 \le t \le T$, then (2.24) and (2.25) are satisfied and hence (3.13),(3.14) has a unique solution $Z \in \bigcap_{k=0}^{3} C^{3-k}([0,T];\mathscr{X}_k)$, by (a-5). Let S be the map which carries w into the function defined on $(-\infty,T]$ by

$$(3.15)\qquad (Sw)(t) = \begin{cases} v(t), & t \le 0 \\ Z(t), & 0 < t \le T \end{cases},$$

where Z is the solution of (3.13),(3.14).

Our goal is to show that S has a unique fixed point in $\mathscr{G}(M,T)$, for appropriately chosen M and T, which will obviously be a solution to the history value problem (2.15),(2.16). The existence of such a fixed point will be established by the contraction mapping principle. For the convenience of the reader, we record below certain inequalities which will be used, without explicit reference, in the subsequent estimates.

If w belongs to $\mathscr{G}(M,T)$, then

$$(3.16)\qquad \operatorname*{ess\text{-}sup}_{t\in[0,T]} \left|\left|\left| \overset{(k)}{w}{}^{t} \right|\right|\right|^2_{3-k} \le 2M^2 + \left|\left|\left| \overset{(k)}{v}{}^{0} \right|\right|\right|^2_{3-k}$$

for each $k=0,1,2,3$,

$$(3.17)\qquad \| \overset{(k)}{w}(t)\|^2_{2-k} \le 2\| \overset{(k)}{v}(0) \|^2_{2-k} + 2t^2M^2, \quad 0 \le t \le T,$$

for each $k=0,1,2$, and

$$(3.18)\qquad \left|\left|\left| \overset{(k)}{w}{}^{t} \right|\right|\right|^2_{2-k} \le 4 \|\overset{(k)}{v}(0)\|^2_{2-k} + 4t^2M^2 + \left|\left|\left| \overset{(k)}{v}{}^{0} \right|\right|\right|^2_{2-k}, \quad 0 \le t \le T,$$

for each $k=0,1,2$.

<u>Lemma 3.2</u>: For M sufficiently large and T sufficiently small, S maps $\mathcal{Z}(M,T)$ into $\mathcal{Z}(M,T)$.

<u>Proof</u>: Take w in $\mathcal{Z}(M,T)$ and let $u = Sw$. In view of the compatibility condition (3.4), u has the requisite smoothness across $t=0$, and by the definition of S, $u(t) = v(t)$ for $t \le 0$. It remains to show that if M is large and T is small, we have

$$(3.19)\qquad \operatorname*{ess\text{-}sup}_{t\in[0,T]} \sum_{k=0}^{3} \| \overset{(k)}{u}(t)\|^2_{3-k} \le M^2,$$

independently of our choice of w.

To simplify the notation, we set

$$(3.20)\qquad b(t) = \int_0^\infty B(w^t,s)w^t(s)ds.$$

Then, u satisfies

$$(3.21)\qquad \ddot{u}(t) + \Lambda(w^t)u(t) = f(t)-b(t), \quad 0 \le t \le T,$$

$$u^{(k)}(0) = v^{(k)}(0), \quad k=0,1,2,3. \tag{3.22}$$

Differentiation of (3.21) with respect to t yields

$$u^{(3)}(t) + A(w^t)\dot{u}(t) = \dot{f}(t)-\dot{b}(t) - A'(w^t;\dot{w}^t)u(t), \quad 0 \leq t \leq T. \tag{3.23}$$

We apply the forward difference operator Δ_η of stepsize η to both sides of (3.23), take the inner product of the resulting expression with $(\Delta_\eta \ddot{u})(t)$, and integrate from 0 to τ. After certain integrations by parts, we divide by η^2 and let η tend to zero. The result of this tedious, yet straightforward computation is

$$\begin{aligned}
\tag{3.24} \frac{1}{2}\| u^{(3)}(\tau)\|_0^2 &+ \frac{1}{2} \langle A(w^\tau)\ddot{u}(\tau),\ddot{u}(\tau)\rangle \\
&= \frac{1}{2} \| v^{(3)}(0)\|_0^2 + \frac{1}{2} \langle A(v^0)\ddot{v}(0),\ddot{v}(0)\rangle \\
&+ \frac{1}{2}\int_0^\tau \langle A'(w^t;\dot{w}^t)\ddot{u}(t),\ddot{u}(t)\rangle \, dt \\
&+ \frac{1}{2}\int_0^\tau a(w^t; u^{(3)}(t),\ddot{u}(t))dt \\
&- 2\int_0^\tau \langle A'(w^t;\dot{w}^t)\dot{u}(t), u^{(3)}(t)\rangle \, dt \\
&- \int_0^\tau \langle A'(w^t;\ddot{w}^t)u(t), u^{(3)}(t)\rangle \, dt
\end{aligned}$$

$$- \int_0^\tau \langle A''(w^t;\dot{w}^t,\dot{w}^t)u(t), \overset{(3)}{u}(t)\rangle \, dt$$

$$+ \int_0^\tau \langle \ddot{f}(t), \overset{(3)}{u}(t)\rangle \, dt$$

$$- \int_0^\tau \langle \ddot{b}(t), \overset{(3)}{u}(t)\rangle \, dt, \qquad 0 \le \tau \le T.$$

From (3.24), we deduce that

$$(3.25) \qquad \|\overset{(3)}{u}(\tau)\|_0^2 + \lambda\|\ddot{u}(\tau)\|_1^2$$

$$\le \|\overset{(3)}{v}(0)\|_0^2 + \langle A(v^0)\ddot{v}(0),\ddot{v}(0)\rangle$$

$$+ \int_0^\tau \|\ddot{f}(t)\|_0^2 dt + \int_0^\tau \|\ddot{b}(t)\|_0^2$$

$$+ P(M) \sum_{k=0}^{3} \int_0^\tau \|\overset{(k)}{u}(t)\|_{3-k}^2 \, dt, \qquad 0 \le \tau \le T,$$

where $P:[0,\infty) \to [0,\infty)$ is a locally bounded function which can be chosen independently of w and T, and λ is a positive (coercivity) constant which is independent of w, M, and T.

Applying $(A(w^t))^{-1}$ to both sides of (3.21) and (3.23), we get

$$(3.26) \qquad u(t) = (A(w^t))^{-1}[f(t)-b(t)-\ddot{u}(t)], \qquad 0 \le t \le T$$

and

$$(3.27)\qquad \dot{u}(t) = (A(w^t))^{-1}[\dot{f}(t)-\dot{b}(t)-\overset{(3)}{u}(t)-A'(w^t;\dot{w}^t)u(t)],$$

$$0 \leq t \leq T,$$

from which it follows that

$$(3.28)\qquad \|u(t)\|_3^2 \leq \mu(\|f(t)\|_1^2 + \|b(t)\|_1^2 + \|\ddot{u}(t)\|_1^2),$$

$$0 \leq t \leq T,$$

and

$$(3.29)\qquad \|\dot{u}(t)\|_2^2 \leq \mu(\|\dot{f}(t)\|_0^2 + \|\dot{b}(t)\|_0^2 + \|\overset{(3)}{u}(t)\|_0^2$$

$$+ (1+M^2t^2)\|u(t)\|_3^2), \quad 0 \leq t \leq T,$$

for some positive constant μ which can be chosen independently of w, M, and T.

Now, set

$$(3.30)\qquad V(w,M,T) = \sup_{t\in[0,T]} \sum_{k=0}^{3} \|\overset{(k)}{u}(t)\|_{3-k}^2,$$

and observe that V can be dominated by a linear combination of the suprema of the left hand sides of (3.25),(3.28), and (3.29). We want to show that if M is sufficiently large and T is sufficiently small, then $V(w,M,T) \leq M^2$, independently of our original choice of w.

In order to secure such a bound for V, we need to estimate the terms involving b which appear on the right hand sides of (3.25),(3.28), and (3.29). Making use of the identities

$$(3.31)\qquad b(t) = \int_0^t B(w^t, t-\xi)w(\xi)d\xi + \int_{-\infty}^0 B(w^t, t-\xi)v(\xi)d\xi,$$

$$(3.32)\qquad \dot{b}(t) = \int_0^t B(w^t, t-\xi)\dot{w}(\xi)d\xi + \int_0^t B'(w^t;\dot{w}^t, t-\xi)w(\xi)d\xi + \int_{-\infty}^0 B(w^t, t-\xi)\dot{v}(\xi)d\xi + \int_{-\infty}^0 B'(w^t;\dot{w}^t, t-\xi)v(\xi)d\xi,$$

and

$$(3.33)\qquad \ddot{b}(t) = B(w^t, 0)\dot{w}(t) + \int_0^\infty \dot{B}(w^t, s)\dot{w}^t(s)ds + 2\int_0^\infty B'(w^t;\dot{w}^t, s)\dot{w}^t(s)ds + \int_0^\infty B'(w^t;\ddot{w}^t, s)w^t(s)ds + \int_0^\infty B''(w^t;\dot{w}^t, \dot{w}^t, s)w^t(s)ds,$$

we deduce that

$$\|b(t)\|_1^2 \le \Lambda + t^2 Q(M), \qquad 0 \le t \le T, \tag{3.34}$$

$$\|\dot{b}(t)\|_0^2 \le \Lambda + t^2 Q(M), \qquad 0 \le t \le T, \tag{3.35}$$

and

$$\|\ddot{b}(t)\|_0^2 \le Q(M), \qquad 0 \le t \le T, \tag{3.36}$$

where $Q:[0,\infty) \to [0,\infty)$ is a locally bounded function which can be chosen independently of w and T, and Λ is a positive constant which is independent of w, M, and T.

Combining (3.25),(3.28),(3.29),(3.34),(3.35), and (3.36) in a straightforward fashion, we arrive at an estimate of the form

$$V(w,M,T) \le \alpha + (T+T^2)R(M) + (T+T^2)R(M)V(w,M,T), \tag{3.37}$$

where $R:[0,\infty) \to [0,\infty)$ is a locally bounded function which can be chosen independently of w and T, and α is a positive constant which is independent of w, M, and T. (Of course, α and R depend on certain properties of f and v.) If we fix M_0 large enough so that $M_0^2 \ge 3\alpha$ and then select T_0 small enough so that $(T_0+T_0^2)R(M_0) \le \alpha$, we deduce from (3.37) that

$$V(w,M_0,T_0) \le M_0^2, \tag{3.38}$$

and the proof is complete. ■

Fix $\bar{M} > 0$ and $\bar{T} > 0$ such that S maps $\mathscr{Z}(\bar{M},T)$ into $\mathscr{Z}(\bar{M},T)$ for every T which satisfies $0 < T \le \bar{T}$.

<u>Lemma 3.3</u>: For T sufficiently small, the map $S: \mathscr{G}(\bar{M},T) \to \mathscr{G}(\bar{M},T)$ is a strict contraction with respect to the metric ρ.

<u>Proof</u>: Take w_1, w_2 in $\mathscr{G}(\bar{M},T)$ where $0 < T \le \bar{T}$, and set $u_1 = Sw_1$, $u_2 = Sw_2$, $U = u_1 - u_2$, $W = w_1 - w_2$. Then, U satisfies

$$(3.39) \quad \ddot{U}(t) + A(w_1^t)U(t) = [A(w_2^t) - A(w_1^t)]u_2(t) + \int_0^\infty [B(w_2^t,s) - B(w_1^t,s)]w_1^t(s)ds - \int_0^\infty B(w_2^t,s)W^t(s)ds, \quad 0 \le t \le T,$$

$$(3.40) \quad U(t) = 0, \quad t \le 0.$$

Differentiating (3.39) with respect to t and rearranging certain terms, we get

$$(3.41) \quad \overset{(3)}{U}(t) + A(w_1^t)\dot{U}(t) = [A(w_2^t) - A(w_1^t)]\dot{u}_2(t) - A'(w_1^t;\dot{w}_1^t)U(t) - A'(w_2^t;\dot{W}^t)u_2(t) + [A'(w_2^t;\dot{w}_1^t) - A'(w_1^t;\dot{w}_1^t)]u_2(t) + \int_0^\infty [B(w_2^t,s) - B(w_1^t,s)]\dot{w}_1^t(s)ds - \int_0^\infty B(w_2^t,s)\dot{W}^t(s)ds - \int_0^\infty B'(w_2^t;\dot{w}_2^t,s)W^t(s)ds$$

$$- \int_0^\infty B'(w_2^t;\dot{W}^t,s)w_1^t(s)ds$$

$$+ \int_0^\infty [B'(w_2^t;\dot{w}_1^t,s)-B'(w_1^t;\dot{w}_1^t,s)]w_1^t(s)ds$$

$$0 \leq t \leq T.$$

We take the inner product of both sides of (3.41) with $\ddot{U}(t)$ and integrate from 0 to τ . After certain integrations by parts, we arrive at

$$(3.42) \qquad \frac{1}{2}\|\ddot{U}(\tau)\|_0^2 + \frac{1}{2} <A(w_1^\tau)\dot{U}(\tau),\dot{U}(\tau)>$$

$$= \frac{1}{2}\int_0^\tau <A'(w_1^t;\dot{w}_1^t)\dot{U}(t),\dot{U}(t)> dt$$

$$+ \frac{1}{2}\int_0^\tau a(w_1^t;\ddot{U}(t),\dot{U}(t))dt$$

$$+ \int_0^\tau <[A(w_2^t)-A(w_1^t)]\dot{u}_2(t),\ddot{U}(t)> dt$$

$$- \int_0^\tau <A'(w_1^t;\dot{w}_1^t)U(t),\ddot{U}(t)> dt$$

$$- \int_0^\tau <A'(w_2^t;\dot{W}^t)u_2(t),\ddot{U}(t)> dt$$

$$+ \int_0^\tau <[A'(w_2^t;\dot{w}_1^t)-A'(w_1^t;\dot{w}_1^t)]u_2(t),\ddot{U}(t)> dt$$

$$+ \int_0^\tau <\int_0^\infty [B(w_2^t,s)-B(w_1^t,s)]\dot{w}_1^t(s)ds,\ddot{U}(t)> dt$$

$$- \int_0^\tau <\int_0^\infty B(w_2^t,s)\dot{W}^t(s)ds,\ddot{U}(t)> dt$$

$$- \int_0^\tau <\int_0^\infty B'(w_2^t;\dot{w}_2^t,s)W^t(s)ds,\ddot{U}(t)> dt$$

$$- \int_0^\tau <\int_0^\infty B'(w_2^t;\dot{W}^t,s)w_1^t(s)ds,\ddot{U}(t)> dt$$

$$+ \int_0^\tau <\int_0^\infty [B'(w_2^t;\dot{w}_1^t,s)-B'(w_1^t;\dot{w}_1^t,s)]w_1^t(s)ds,\ddot{U}(t)> dt,$$

$$0 \le t \le T.$$

Since $W(t) = 0$ for $t \le 0$, it follows that

$$(3.43) \qquad |||W^t|||_2^2 \le 2 \sup_{t\in[0,T]} \|W(t)\|_2^2, \qquad 0 \le t \le T,$$

and

$$(3.44) \qquad |||\dot{W}^t|||_1^2 \le 2 \sup_{t\in[0,T]} \|\dot{W}(t)\|_1^2, \qquad 0 \le t \le T.$$

Thus from (3.42), we deduce that

$$(3.45) \qquad \|\ddot{U}(\tau)\|_0^2 + \lambda\|\ddot{U}(\tau)\|_1^2 \le \mu[\sum_{k=0}^{2} \int_0^\tau \|\overset{(k)}{U}(t)\|_{2-k}^2 \, dt$$

$$+ \tau \sup_{t\in[0,T]} (\|W(t)\|_2^2 + \|\dot{W}(t)\|_1^2)], \quad 0 \le \tau \le T,$$

where λ and μ are positive constants which can be chosen independently of w_1, w_2, and T.

Applying $(A(w^t))^{-1}$ to both sides of (3.39) and making use of the fact that $W^t(s) = 0$ for $s \ge t$, we get

$$U(t) = (A(w_1^t))^{-1}\{[A(w_2^t)-A(w_1^t)]u_2(t) - \dddot{U}(t) - \int_0^t B(w_2^t,s)W^t(s)ds - \int_0^\infty [B(w_2^t,s)-B(w_1^t,s)]w_1^t(s)ds\}, \quad 0 \le t \le T. \tag{3.46}$$

It follows from (3.46) and the inequality

$$|||W^t|||_1^2 \le 2(\int_0^T \|\dot{W}(\xi)\|_1 d\xi)^2, \qquad 0 \le t \le T, \tag{3.47}$$

that

$$\|U(t)\|_2^2 \le \gamma[\|\ddot{U}(t)\|_0^2 + T^2 \sup_{\xi\in[0,T]} (\|W(\xi)\|_2^2 + \|\dot{W}(\xi)\|_1^2)], \qquad 0 \le t \le T, \tag{3.48}$$

where γ is a positive constant which is independent of w_1, w_2, and T.

Combining (3.45) and (3.48), we obtain an estimate of the form

$$\sup_{t\in[0,T]} \sum_{k=0}^{2} \|\overset{(k)}{U}(t)\|_{2-k}^2 \le \alpha(T+T^2)\{\sup_{t\in[0,T]} \sum_{k=0}^{2} \|\overset{(k)}{U}(t)\|_{2-k}^2 + \sup_{t\in[0,T]} \sum_{k=0}^{2} \|\overset{(k)}{W}(t)\|_{2-k}^2\}, \tag{3.49}$$

where α is a positive constant which can be chosen independently of w_1, w_2, and T. (Of course, α depends on ν and $\overline{M}$.) If we select T small enough so that $3\alpha T(T+T^2) \leq 1$, then (3.49) yields

$$(3.50) \qquad \rho(sw_1, sw_2) \leq \frac{1}{2}\rho(w_1, w_2)$$

$$\forall\, w_1, w_2 \in \mathscr{Y}(\bar{M}, T),$$

which completes the proof. ■

Remark 3.1: The proofs of Lemmas 3.1, 3.2, and 3.3 remain valid if we drop the compatibility assumption (3.5) and replace (3.3) with the weaker condition

$$(3.51) \qquad v \in \bigcap_{k=0}^{3} W^{3-k,\infty}((-\infty,0];\mathscr{X}_k), \quad \overset{(k)}{v}(0) \in \mathscr{X}_{3-k},$$

$$k=0,1,2.$$

Proof of Theorem 3.1: From Lemmas 3.1, 3.2, 3.3, and the contraction mapping principle, we deduce that S has a unique fixed point in $\mathscr{Y}(M,T)$ for sufficiently large M and sufficiently small $T > 0$, which is the unique solution of (2.15),(2.16) in $\bigcap_{k=0}^{3} W^{3-k,\infty}((-\infty,T];\mathscr{X}_k)$. Let $(-\infty,\bar{t})$ be the maximal interval on which a solution u of (2.15),(2.16) exists and satisfies

$u \in \bigcap_{k=0}^{3} W^{3-k,\infty}((-\infty,T];\mathscr{X}_k)$ for every $T < \bar{t}$.

Suppose that $\bar{t} < \infty$ and that $\operatorname{ess-sup}_{s\in[0,t]} \sum_{k=0}^{3} \| \overset{(k)}{u}(s)\|^2_{3-k}$ remains bounded as $t \to \bar{t}$. Then, we can extend u to be defined on $(-\infty,\bar{t}]$ such that the extended function satisfies

$$(3.52)\qquad u \in \bigcap_{k=0}^{3} W^{3-k,\infty}((-\infty,\bar{t}];\mathscr{X}_k),\quad \overset{(k)}{u}(\bar{t}) \in \mathscr{X}_{3-k},\ k=0,1,2,$$

and

$$(3.53)\qquad \ddot{u}(\bar{t}) = -A(u^{\bar{t}})u(\bar{t}) - \int_0^\infty B(u^{\bar{t}},s)u^{\bar{t}}(s)ds + f(\bar{t}).$$

Making an obvious translation of variables, we can now use Lemmas 3.1, 3.2, 3.3, and the contraction mapping principle again to extend u so that it is a solution on some interval $(-\infty,t^*]$ with $t^* > \bar{t}$, which contradicts the assumption that $(-\infty,\bar{t})$ is maximal. Thus, if $\bar{t} < \infty$, then

$$\operatorname*{ess-sup}_{s\in[0,t]} \sum_{k=0}^{3} \| \overset{(k)}{u}(s)\|^2_{3-k} \to \infty \quad \text{as } t \uparrow \bar{t}.$$

Finally, because of (3.3), (3.5), and that fact that u is a solution of the initial value problem (3.13), (3.14) (with w=u) on [0,T] for each $T < \bar{t}$, we actually have that $u \in \bigcap_{k=0}^{3} C^{3-k}((-\infty,T];\mathscr{X}_k)$ for each $T < \bar{t}$. The proof of Theorem 3.1 is complete. ■

<u>Remark 3.2</u>: If we drop assumption (3.5), Theorem 3.1 remains valid with the exception that $\overset{(3)}{u}$ may be discontinuous at t=0, i.e. the history value problem (2.15),(2.16) has a unique local solution u defined on a maximal interval $(-\infty,T_{max})$, $T_{max} > 0$, such that for each T with $0 < T < T_{max}$, the restriction of u to $(-\infty,T]$ satisfies $u \in \bigcap_{k=1}^{3} C^{3-k}((-\infty,T];\ _k)$ and the restriction of u to [0,T] satisfies $u \in \bigcap_{k=0}^{3} C^{3-k}([0,T];\mathscr{X}_k)$, and if $T_{max} < \infty$, then (3.7) holds.

Chapter 4. Global Existence

In this chapter we show that the history value problem (2.15),(2.16) has a unique global solution, provided that f and v are suitably small. We assume throughout that the basic assumptions (a-1) through (a-11) hold. We also assume that f satisfies

$$(4.1) \qquad f \in C^0([0,\infty);\mathscr{X}_1) \cap L^2([0,\infty);\mathscr{X}_1),$$

$$(4.2) \qquad \dot{f} \in C^0([0,\infty);\mathscr{X}_0) \cap L^2([0,\infty);\mathscr{X}_0),$$

$$(4.3) \qquad \ddot{f} \in L^2([0,\infty);\mathscr{X}_0),$$

and that v satisfies

$$(4.4) \qquad v \in \bigcap_{k=0}^{3} C^{3-k}((-\infty,0];\mathscr{X}_k),$$

and the compatibility conditions

$$(4.5) \qquad \ddot{v}(0) = -A(v^0)v(0) - \int_0^\infty B(v^0,s)v^0(s)ds + f(0),$$

$$(4.6) \qquad \overset{(3)}{v}(0) = -A(v^0)\dot{v}(0) - A'(v^0;\dot{v}^0)v(0) - \int_0^\infty B(v^0,s)\dot{v}^0(s)ds - \int_0^\infty B'(v^0;\dot{v}^0,s)v^0(s)ds + \dot{f}(0).$$

The sole purpose of (4.5),(4.6) is to ensure that the solution will be smooth across t=0. (See Remark 4.3.)

As discussed in the Introduction a global solution should be expected only when f and v are suitably small. To measure the sizes of f and v, we define

$$(4.7)\qquad \mathscr{F}(f) = \sup_{t\in[0,\infty)} (\|f(t)\|_1^2 + \|\dot{f}(t)\|_0^2) + \int_0^\infty (\|f(t)\|_1^2 + \|\dot{f}(t)\|_0^2 + \|\ddot{f}(t)\|_0^2)dt,$$

and

$$(4.8)\qquad \mathscr{V}(v) = \sum_{k=0}^{2} \| \overset{(k)}{v}{}^0 \|_{3-k}^2 .$$

Our main result is:

Theorem 4.1: Assume that the basic assumptions (a-1) through (a-11) hold and that the influence function h satisfies

$$(4.9)\qquad h(t+s) \le ch(t)h(s), \quad \forall\ t,s \ge 0,$$

for some positive constant c. Then, there is a positive constant δ such that for each f and v which satisfy (4.1) through (4.6) with

$$(4.10)\qquad \mathscr{F}(f) + \mathscr{V}(v) \le \delta ,$$

the history value problem (2.15),(2.16) has a unique solution $u \in \bigcap_{k=0}^{3} C^{3-k}((-\infty,\infty);\mathscr{X}_k)$ and

$$(4.11)\qquad \overset{(k)}{u}(t) \to 0 \ \text{ in } \mathscr{X}_{3-k-1}, \text{ as } t \to \infty,$$

for k=0,1,2. Moreover there is a positive constant Γ such that

$$(4.12) \qquad \sum_{k=0}^{3} \| \overset{(k)}{u}(t) \|_{3-k}^{2} \le \Gamma\{ \mathcal{F}(f) + \mathcal{V}\, v) \} \qquad \forall\ t \ge 0.$$

Remark 4.1: Clearly, exponential influence functions of the form $h(s) = Me^{-ds}$, $M > 0$, $d > 0$ satisfy (4.9). However, influence functions of the form $h(s) = M(1+s)^{-d}$, $M > 0$, $d > 1$, do not satisfy (4.9).

Remark 4.2: Theorem 4.1 remains valid if we drop the assumption (4.9) and replace (4.10) with the stronger condition

$$(4.13) \qquad \mathcal{F}(f) + \mathcal{V}(v) + \int_0^\infty \int_0^\infty \sum_{k=0}^{2} h(t+s) \| \overset{(k)}{v}{}^{0}(s) \|_{3-k}^{2}\, ds dt \le \delta.$$

The proof requires only trivial modifications.

Remark 4.3: If we drop the compatibility assumption (4.6), Theorem 4.1 remains valid with the exception that $\overset{(3)}{u}$ may be discontinuous at t=0, i.e. if $\mathcal{F}(f) + \mathcal{V}(v) \le \delta$, then (2.15), (2.16) has a unique solution $u \in \bigcap_{k=1}^{3} C^{3-k}((-\infty,\infty); \mathcal{X}_k)$ such that the restriction of u to $[0,\infty)$ satisfies $u \in \bigcap_{k=0}^{3} C^{3-k}([0,\infty); \mathcal{X}_k)$, and (4.11) and (4.12) hold.

Remark 4.4: It is interesting to observe that the symmetry condition (a-4) on A is not used to derive any of the global estimates in the proof of Theorem 4.1. It is, however, needed as an assumption in Theorem 4.1 because it is used in the local existence proof.

Proof of Theorem 4.1: Clearly the assumptions of Theorem 3.1 are satisfied, so let u be the solution of (2.15),(2.16) on a maximal interval $(-\infty, T_{max})$. For $\tau \in [0, T_{max})$, set

$$\mathscr{E}(u(\tau)) = \sup_{t\in[0,\tau]} \sum_{k=0}^{3} \| \overset{(k)}{u}(t)\|_{3-k}^{2} + \sum_{k=0}^{3} \int_0^{\tau} \| \overset{(k)}{u}(t)\|_{3-k}^{2} . \tag{4.14}$$

Our goal is to show that if (4.9) is satisfied with δ sufficiently small, then

$$\mathscr{E}(u(\tau)) \leq \kappa\{\mathscr{F}(f) + \mathscr{V}(v)\}, \quad 0 \leq \tau < T_{max} , \tag{4.15}$$

for some positive constant κ which can be chosen independently of δ, f, and v. This will obviously imply $T_{max} = \infty$ by Theorem 3.1 and yield the estimate (4.12). Also, (4.15) will imply that the restriction of u to $[0,\infty)$ satisfies

$$u \in \bigcap_{k=0}^{3} W^{3-k,2}([0,\infty);\mathscr{X}_k), \tag{4.16}$$

from which (4.11) follows immediately.

To establish (4.15), we develop a chain of energy estimates. We begin by rewriting (2.15) in the equivalent form

$$\begin{aligned} \ddot{u}(t) + F(0)u(t) + \int_0^{\infty} C(0,s)\dot{u}^t(s)ds \\ = f(t) + [A(0)-A(u^t)]u(t) \\ + \int_0^{\infty} [B(0,s)-B(u^t,s)]u^t(s)ds, \end{aligned} \tag{4.17}$$

which in turn is equivalent to

$$\ddot{u}(t) + F(0)u(t) + \int_0^t C(0,t-\xi)\dot{u}(\xi)d\xi \tag{4.18}$$

$$= f(t) + [A(0)-A(u^t)]u(t)$$

$$+ \int_0^\infty [B(0,s)-B(u^t,s)]u^t(s)ds$$

$$- \int_{-\infty}^0 C(0,t-\xi)\dot{v}(\xi)d\xi.$$

We take the inner product of $C(0,0)\dot{u}(t)$ with both sides of (4.18) and integrate from 0 to τ, thus obtaining

$$\frac{1}{2} \langle C(0,0)\dot{u}(\tau),\dot{u}(\tau)\rangle \tag{4.19}$$

$$+ \frac{1}{2} \langle C(0,0)u(\tau),F(0)u(\tau)\rangle$$

$$+ \int_0^\tau \langle C(0,0)\dot{u}(t), \int_0^t C(0,t-\xi)\dot{u}(\xi)d\xi\rangle\, dt$$

$$= \frac{1}{2} \langle C(0,0)\dot{v}(0),\dot{v}(0)\rangle$$

$$+ \frac{1}{2} \langle C(0,0)v(0), F(0)v(0)\rangle$$

$$+ \int_0^\tau \langle C(0,0)\dot{u}(t),f(t)\rangle\, dt$$

$$+ \int_0^\tau \langle C(0,0)\dot{u}(t),[A(0)-A(u^t)]u(t)\rangle\, dt$$

$$+ \int_0^\tau \langle C(0,0)\dot{u}(t), \int_0^\infty [B(0,s)-B(u^t,s)]u^t(s)ds\rangle\, dt$$

$$- \int_0^{\tau} \langle C(0,0)\dot{u}(t), \int_{-\infty}^{0} C(0,t-\xi)\dot{v}(\xi)d\xi\rangle \, dt,$$

$$0 \le \tau < T_{max}.$$

Next, we apply the forward difference operator Δ_η of stepsize η to both sides of (4.17). After making use of the identity

$$(4.20) \qquad \int_0^{\infty} C(0,s)(\Delta_\eta \dot{u})^t(s) = \int_0^{t} C(0,t-\xi)(\Delta_\eta \dot{u})(\xi)d\xi + \int_{-\infty}^{0} C(0,t-\xi)(\Delta_\eta \dot{v})(\xi)d\xi,$$

we take the inner product of the resulting expression with $C(0,0)(\Delta_\eta \dot{u})(t)$ and integrate from 0 to τ. After numerous integrations by parts, we divide by η^2 and let η tend to zero. The outcome of this computation is

$$(4.21) \qquad \frac{1}{2}\langle C(0,0)\ddot{u}(\tau),\ddot{u}(\tau)\rangle + \frac{1}{2}\langle C(0,0)\dot{u}(\tau),F(0)\dot{u}(\tau)\rangle$$
$$+ \lim_{\eta\downarrow 0} \frac{1}{\eta^2}\int_0^{\tau} \langle C(0,0)(\Delta_\eta \dot{u})(t), \int_0^{t} C(0,t-\xi)(\Delta_\eta \dot{u})(\xi)d\xi\rangle \, dt$$
$$= \frac{1}{2}\langle C(0,0)\ddot{v}(0),\ddot{v}(0)\rangle + \frac{1}{2}\langle C(0,0)\dot{v}(0),F(0)\dot{v}(0)\rangle$$
$$+ \langle C(0,0)\dot{u}(\tau),\dot{f}(\tau)\rangle - \langle C(0,0)\dot{v}(0),\dot{f}(0)\rangle$$
$$- \int_0^{\tau} \langle C(0,0)\dot{u}(t),\ddot{f}(t)\rangle \, dt$$
$$+ \frac{1}{2}\langle C(0,0)\dot{u}(\tau),[A(0)-A(u^{\tau})]\dot{u}(\tau)\rangle$$

$$+ \frac{1}{2} \langle C(0,0)\dot{v}(0), [A(v^0) - A(0)]\dot{v}(0)\rangle$$

$$+ \frac{1}{2} \int_0^\tau \langle C(0,0)\dot{u}(t), A'(u^t;\dot{u}^t)\dot{u}(t)\rangle \, dt$$

$$+ \langle C(0,0)\dot{u}(\tau), \int_0^\infty [B(0,s) - B(u^t,s)]\dot{u}^t(s)ds\rangle$$

$$+ \langle C(0,0)\dot{v}(0), \int_0^\infty [B(v^0,s) - B(0,s)]\dot{v}^0(s)ds\rangle$$

$$+ \int_0^\tau \langle C(0,0)\dot{u}(t), \int_0^\infty B'(u^t;\dot{u}^t,s)\dot{u}^t(s)ds\rangle \, dt$$

$$+ \int_0^\tau \langle C(0,0)\dot{u}(t), [B(u^t,0) - B(0,0)]\dot{u}(t)\rangle \, dt$$

$$+ \int_0^\tau \langle C(0,0)\dot{u}(t), \int_0^\infty [\dot{B}(u^t,s) - \dot{B}(0,s)]\dot{u}^t(s)ds\rangle \, dt$$

$$- \langle C(0,0)\dot{u}(\tau), A'(u^\tau;\dot{u}^\tau)u(\tau)\rangle$$

$$+ \langle C(0,0)\dot{v}(0), A'(v^0;\dot{v}^0)v(0)\rangle$$

$$+ \int_0^\tau \langle C(0,0)\dot{u}(t), A'(u^t;\dot{u}^t)\dot{u}(t)\rangle \, dt$$

$$+ \int_0^\tau \langle C(0,0)\dot{u}(t), A'(u^t;\ddot{u}^t)u(t)\rangle \, dt$$

$$+ \int_0^\tau \langle C(0,0)\dot{u}(t), A''(u^t;\dot{u}^t,\dot{u}^t)u(t)\rangle \, dt$$

$$- \langle C(0,0)\dot{u}(\tau), \int_0^\infty B'(u^\tau;\dot{u}^\tau,s)u^\tau(s)ds\rangle$$

$$+ \langle C(0,0)\dot{v}(0), \int_0^\infty B'(v^0;\dot{v}^0,s)v^0(s)ds\rangle$$

$$+ \int_0^\tau \langle C(0,0)\dot{u}(t), \int_0^\infty B'(u^t;\dot{u}^t,s)\dot{u}^t(s)ds\rangle\, dt$$

$$+ \int_0^\tau \langle C(0,0)\dot{u}(t), \int_0^\infty B'(u^t;\ddot{u}^t,s)u^t(s)ds\rangle\, dt$$

$$+ \int_0^\tau \langle C(0,0)\dot{u}(t), \int_0^\infty B''(u^t;\dot{u}^t,\dot{u}^t,s)u^t(s)ds\rangle\, dt$$

$$- \langle C(0,0)\dot{u}(\tau),C(0,\tau)\dot{v}(0)\rangle + \langle C(0,0)\dot{v}(0),C(0,0)\dot{v}(0)\rangle$$

$$+ \int_0^\tau \langle C(0,0)\dot{u}(t),B(0,t)\dot{v}(0)\rangle\, dt$$

$$- \langle C(0,0)\dot{u}(\tau), \int_{-\infty}^0 B(0,\tau-\xi)\dot{v}(\xi)ds\rangle$$

$$+ \langle C(0,0)\dot{v}(0), \int_{-\infty}^0 B(0,-\xi)\dot{v}(\xi)ds\rangle$$

$$+ \int_0^\tau \langle C(0,0)\dot{u}(t), \int_{-\infty}^0 \dot{B}(0,t-\xi)\dot{v}(\xi)ds\rangle\, dt,$$

$$0 \leq \tau < T_{max} .$$

It is not a priori evident that

$$\lim_{\eta \downarrow 0} \frac{1}{\eta^2} \int_0^\tau \langle C(0,0)(\Delta_\eta \dot{u})(t), \int_0^t C(0,t-\xi)(\Delta_\eta \dot{u})(\xi)d\xi\rangle\, dt \quad \text{exists.}$$

However, the limit of each of the other terms involved in the derivation of (4.21) exists, so consequently the limit in question also exists (and is finite) for each $\tau \in [0, T_{max})$. Moreover, from (2.43), we conclude that

$$\lim_{\eta \downarrow 0} \frac{1}{\eta^2} \int_0^\tau \langle C(0,0)(\Delta_\eta \dot{u})(t), \int_0^t C(0,t-\xi)(\Delta_\eta \dot{u})(\xi)d\xi \rangle dt \geq 0. \tag{4.22}$$

Differentiation of (2.15) with respect to t yields

$$\overset{(3)}{u}(t) + A(u^t)\dot{u}(t) + A'(u^t;\dot{u}^t)u(t) + \int_0^\infty B(u^t,s)\dot{u}^t(s)ds + \int_0^\infty B'(u^t;\dot{u}^t,s)u^t(s)ds = \dot{f}(t), \tag{4.23}$$

which can be rewritten as

$$\overset{(3)}{u}(t) + A(u^t)\dot{u}(t) + A(u^t;\dot{u}^t)u(t) + \int_0^t B(u^t,t-\xi)\dot{u}(\xi)d\xi + \int_{-\infty}^0 B(u^t,t-\xi)\dot{v}(\xi)d\xi + \int_0^\infty B'(u^t;\dot{u}^t,s)u^t(s)ds = \dot{f}(f). \tag{4.24}$$

From (4.24), we easily deduce

$$\begin{aligned} \|\overset{(3)}{u}(t)\|_0^2 - 6\|A(u^t)\dot{u}(t)\|_0^2 - 6\|\int_0^t B(u^t,t-\xi)\dot{u}(\xi)d\xi\|_0^2 &\leq 6\|A'(u^t;\dot{u}^t)u(t)\|_0^2 + 6\|\int_{-\infty}^0 B(u^t,t-\xi)\dot{v}(\xi)d\xi\|_0^2 \\ &\quad + 6\|\int_0^\infty B'(u^t;\dot{u}^t)u^t(s)ds\|_0^2 + 6\|\dot{f}(t)\|_0^2, \\ &\qquad 0 \leq t < T_{max}. \end{aligned} \tag{4.25}$$

To obtain our next estimate, we rewrite (2.15) in the form

$$(4.26)\qquad A(0)u(t) + \int_0^t B(0,t-\xi)u(\xi)d\xi = f(t)-\ddot{u}(t)$$
$$+ [A(0)-A(u^t)]u(t)$$
$$+ \int_0^\infty [B(0,s)-B(u^t,s)]u^t(s)ds$$
$$- \int_{-\infty}^0 B(0,t-\xi)v(\xi)d\xi.$$

Then, by using (a-11), we arrive at

$$(4.27)\qquad \sup_{t\in[0,\tau]} \|u(t)\|_3^2 - \mu \sup_{t\in[0,\tau]} \|\ddot{u}(t)\|_1^2$$
$$\leq \mu \sup_{t\in[0,\tau]} \|f(t)\|_1^2 + \mu \sup_{t\in[0,\tau]} \|[A(0)-A(u^t)]u(t)\|_1^2$$
$$+ \mu \sup_{t\in[0,\tau]} \left\|\int_0^\infty [B(0,s)-B(u^t,s)]u^t(s)ds\right\|_1^2$$
$$+ \mu \sup_{t\in[0,\tau]} \left\|\int_{-\infty}^0 B(0,t-\xi)v(\xi)d\xi\right\|_1^2, \quad 0 \leq \tau < T_{max},$$

where μ is a positive constant which is independent of f and v.

Observe now that by combining (4.19),(4.21),(4.25) and (4.27), we can dominate

$$\sup_{t\in[0,\tau]} \sum_{k=0}^{3} \|\overset{(k)}{u}(t)\|_{3-k}^2$$
$$+ \int_0^\tau \langle C(0,0)\dot{u}(t), \int_0^t C(0,t-\xi)\dot{u}(\xi)d\xi\rangle\, dt$$
$$+ \lim_{\eta\downarrow 0} \frac{1}{\eta^2} \int_0^\tau \langle C(0,0)(\Delta_\eta \dot{u})(t), \int_0^t C(0,t-\xi)(\Delta_\eta \dot{u})(\xi)d\xi\rangle\, dt$$

by a linear combination of the suprema over $[0,\tau]$ of the right hand sides of (4.19),(4.21),(4.25), and (4.27). The last two terms in the above expression (which are positive by (a-10)) represent the "dissipative contribution" of the memory and will play the crucial role in securing global existence. We want to take advantage of these terms by using them to derive estimates for $\sum_{k=0}^{3} \int_0^\tau \| \overset{(k)}{u}(t) \|_{3-k}^2 \, dt$.

To this end, we apply the Cauchy inequality to the identity

$$(4.28) \qquad C(0,0)(\Delta_\eta u)(t) = C(0,t)(\Delta_\eta u)(0) + \int_0^t C(0,t-\xi)(\Delta_\eta \dot{u})(\xi)d\xi - \int_0^t B(0,t-\xi)(\Delta_\eta u)(\xi)d\xi,$$

and integrate from 0 to τ. We then use (a-10), divide by η^2 and let η tend to zero to obtain

$$(4.29) \qquad \int_0^\tau \|C(0,0)\dot{u}(t)\|_0^2 \, dt - 3\beta \int_0^\tau \langle C(0,0)\dot{u}(t), \int_0^t C(0,t-\xi)\dot{u}(\xi)d\xi \rangle \, dt - 3\beta \lim_{\eta \downarrow 0} \frac{1}{\eta^2} \int_0^\tau \langle C(0,0)(\Delta_\eta \dot{u})(t), \int_0^t C(0,t-\xi)(\Delta_\eta \dot{u})(\xi)d\xi \rangle \, dt \le 3 \int_0^\infty \|C(0,t)\dot{v}(0)\|_0^2 \, dt, \qquad 0 \le \tau < T_{max},$$

where β is a positive constant which can be chosen independently of f and v.

From (4.26) and (a-11), we easily get

$$(4.30) \qquad \int_0^\tau \|u(t)\|_3^2\,dt - \gamma \int_0^\tau \|\ddot{u}(t)\|_1^2\,dt \le \gamma \int_0^\tau \|f(t)\|_1^2\,dt$$

$$+ \gamma \int_0^\tau \|[A(0)-A(u^t)]u(t)\|_1^2\,dt$$

$$+ \gamma \int_0^\tau \Big\|\int_0^\infty [B(0,s)-B(u^t,s)]u^t(s)ds\Big\|_1^2\,dt$$

$$+ \gamma \int_0^\tau \Big\|\int_{-\infty}^0 B(0,t-\xi)v(\xi)d\xi\Big\|_1^2\,dt, \qquad 0 \le \tau < T_{max},$$

for some positive constant γ which is independent of f and v.

Our next estimate is obtained by the following procedure: We apply the forward difference operator Δ_η to both sides of (4.17) and make use of (4.20). After applying the Cauchy inequality, we integrate from 0 to τ and use (a-10). Then, we divided by η^2 and let η tend to zero. The result of this computation is

$$(4.31) \qquad \int_0^\tau \|\overset{(3)}{u}(t)\|_0^2\,dt - 9\int_0^\tau \|F(0)\dot{u}(t)\|_1^2\,dt$$

$$- 9\beta \lim_{\eta\downarrow 0} \frac{1}{\eta^2}\int_0^\tau \Big\langle C(0,0)(\Delta_\eta\dot{u})(t), \int_0^t C(0,t-\xi)(\Delta_\eta\dot{u})(\xi)d\xi\Big\rangle\,dt$$

$$\le 9\int_0^\tau \|\dot{f}(t)\|_0^2\,dt + 9\int_0^\tau \|C(0,t)\dot{v}(0)\|_0^2\,dt$$

$$+ 9\int_0^\tau \Big\|\int_{-\infty}^0 B(0,t-\xi)\dot{v}(\xi)d\xi\Big\|_0^2 + 9\int_0^\tau \|[A(0)-A(u^t)]\dot{u}(t)\|_0^2\,dt$$

$$+ 9 \int_0^\tau \|A'(u^t;\dot{u}^t)u(t)\|_0^2$$

$$+ 9 \int_0^\tau \| \int_0^\infty [B(0,s)-B(u^t,s)]\dot{u}^t(s)ds\|_0^2\, dt$$

$$+ 9 \int_0^\tau \| \int_0^\infty B'(u^t;\dot{u}^t,s)u^t(s)ds\|_0^2\, dt, \qquad 0 \leq \tau < T_{max}.$$

Out final estimate

$$(4.32) \qquad \int_0^\tau <F(0)\ddot{u}(t),\ddot{u}(t)> dt - \frac{1}{2}\int_0^\tau \|F(0)\dot{u}(t)\|_0^2\, dt$$

$$- \frac{1}{2}\int_0^\tau \|\overset{(3)}{u}(t)\|_0^2\, dt - \frac{1}{2}\|F(0)\dot{u}(\tau)\|_0^2 - \frac{1}{2}\|\ddot{u}(\tau)\|_0^2$$

$$\leq \frac{1}{2}\|F(0)\dot{v}(0)\|_0^2 + \frac{1}{2}\|\ddot{v}(0)\|_0^2, \qquad 0 \leq \tau < T_{max},$$

follows easily from the identity

$$(4.33) \qquad \int_0^\tau <F(0)\ddot{u}(t),\ddot{u}(t)> dt = <F(0)\dot{u}(\tau),\ddot{u}(\tau)>$$

$$- <F(0)\dot{v}(0),\ddot{v}(0)> - \int_0^\tau <F(0)\dot{u}(t), \overset{(3)}{u}(t)> dt.$$

Before completing the proof, we pause to comment on the role of (4.9). By combining (4.19),(4.21),(4.25),(4.27),(4.29), (4.30),(4.31), and (4.32), we can bound $\mathscr{E}(u(\tau))$ in terms of the quantities which appear on the right hand sides of these estimates. We want to show that the right hand sides are appropriately "small" if $\mathscr{E}$, $\mathscr{F}$, and $\mathscr{V}$ are small. This will require that the integral

$$\int_0^\tau \int_0^\infty \sum_{k=0}^{2} h(t+s) \| \overset{(k)}{v}{}^0(s) \|^2_{3-k} \, dsdt$$

be small uniformly in $\tau \geq 0$ if $\mathscr{V}$ is small. When h satisfies (4.9), we have

(4.34) $$\sum_{k=0}^{2} \int_0^\tau \int_0^\infty h(t+s) \, \| \overset{(k)}{v}{}^0(s) \|^2_{3-k} \, dsdt$$

$$\leq c \; \mathscr{V}(v), \quad \forall \; \tau \geq 0.$$

If, on the other hand, (4.9) does not hold, we can replace (4.10) with (4.13) and the proof concludes in essentially the same manner.

Observe that by combining (4.19),(4.21),(4.25),(4.27), (4.29),(4.30),(4.31), and (4.32), we can dominate $\mathscr{E}(u(\tau))$ by a linear combination of the suprema over $[0,\tau]$ of the right hand sides of these estimates. Suppose now that

(4.35) $$\mathscr{F}(f) + \mathscr{V}(v) \leq 1,$$

and that

(4.36) $$\mathscr{E}(u(\tau)) \leq \nu^2$$

for some ν with $0 \leq \nu \leq 1$. Then the supremum over $[0,\tau]$ of the absolute value of each term which appears on the right hand side of (4.19),(4.21),(4.25),(4.27),(4.29),(4.30),(4.31), or (4.32) can be majorized by one of $\nu\Lambda\,\mathscr{E}(u(\tau))$, $\Lambda\{\mathscr{F}(f)+\mathscr{V}(v)\}$, or $\varepsilon\Lambda\,\mathscr{E}(u(\tau)) + \frac{\Lambda}{\varepsilon}\{\mathscr{F}(f)+\mathscr{V}(v)\}$ for each $\varepsilon > 0$, where Λ is a positive constant which can be chosen independently of f, v, τ, ν, and ε. Thus if (4.35) and (4.36) hold with $0 \le \cdot \le 1$, we have an estimate of the form

$$(4.37) \qquad \mathscr{E}(u(\tau)) \le (\nu+\varepsilon)M\mathscr{E}(u(\tau)) + M(1+\tfrac{1}{\varepsilon})\{\mathscr{F}(f)+\mathscr{V}(v)\},$$

valid for any $\varepsilon > 0$, where M is a positive constant which is independent of f, v, τ, and ν. If we set $\varepsilon = \frac{1}{4M}$, then (4.37) yields

$$(4.38) \qquad \frac{3}{4}\,\mathscr{E}(u(\tau)) \le \nu M\mathscr{E}(u(\tau)) + (4M^2+M)\{\mathscr{F}(f)+\mathscr{V}(v)\}.$$

It is now evident that if (4.35) and (4.36) are satisfied with $\nu^2 \le \min(1, \frac{1}{4M})$, then

$$(4.39) \qquad \mathscr{E}(u(\tau)) \le (8M^2+M)\{\mathscr{F}(f)+\mathscr{V}(v)\}.$$

Choose $\delta_0 > 0$ such that if $\{\mathscr{F}(f)+\mathscr{V}(v)\} \le \delta_0$, then $\mathscr{E}(u(0)) \le \frac{1}{2}\min(1, \frac{1}{4M})$, and set $\delta_1 = \frac{1}{2}\min(\frac{1}{8M^2+M}, \frac{1}{4M(8M^2+1)})$, $\delta = \min(1,\delta_0,\delta_1)$. (The existence of such a δ_0 follows easily from the definitions of $\mathscr{E}$, $\mathscr{F}$, and $\mathscr{V}$.)

Assume that $\{\mathscr{F}(f)+\mathscr{V}(v)\} \le \delta$. Then, (4.39) implies that there is *no* $\tau \in [0,T_{max})$ for which $\mathscr{E}(u(\tau)) = \min(1, \frac{1}{4M})$. This, in conjunction with the fact that $\mathscr{E}(u(0)) < \min(1, \frac{1}{4M})$ implies that $\mathscr{E}(u(\tau)) < \min(1, \frac{1}{4M})$ for all $\tau \in [0,T_{max})$, and the cycle closes, yielding

$$(4.40) \qquad \mathscr{E}(u(\tau)) \le (8M^2+M)\{\mathscr{F}(f)+\mathscr{V}(v)\} \qquad \forall\ \tau \in [0,T_{max}).$$

Thus, (4.15) is established and the proof is complete. ■

Chapter 5. Materials with Fading Memory.

We now apply the results of the preceding chapters to establish global existence of smooth solutions to the equations of motion for materials with fading memory. Consider the longitudinal motion of a homogeneous one-dimensional body with reference configuration $\mathscr{B} = (0,1)$, a natural state, and unit reference density*. As in the Introduction, we let $u(x,t)$ be the displacement at time t of the particle with reference position x, and we use σ and ε to denote the stress and strain.

We assume that the stress is determined by the temporal history of the strain through a constitutive relation of the form

$$(5.1) \qquad \sigma(x,t) = \mathscr{G}(\varepsilon^t(x,\cdot)),$$

where $\varepsilon^t(x,s) = \varepsilon(x,t-s)$, $s \geq 0$, and $\mathscr{G}$ is a real-valued functional with domain in V_h for some influence function h. Recall that V_h is the set of all measurable functions $w:[0,\infty) \to \mathbb{R}$ such that $\int_0^\infty h(s)|w(s)|^2 ds < \infty$ equipped with the norm given by

$$(5.2) \qquad \|w\|_h^2 = |w(0)|^2 + \int_0^\infty h(s)|w(s)|^2 ds,$$

and that $h \in L^1(0,\infty)$ is assumed to be positive and nonincreasing.

We assume that there is a neighborhood $\mathscr{O}$ of zero in V_h

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* The assumptions of homogeneity and unit density are made only for the sake of simplicity.

such that $\mathscr{G}$ is defined and continuously Fréchet differentiable on $\mathcal{O}$. The Riesz Representation Theorem then implies that the Fréchet derivative $\mathscr{G}'$ of $\mathscr{G}$ admits the representation

$$(5.3) \qquad \mathscr{G}'(w;\bar{w}) = E(w)\bar{w}(0) - \int_0^\infty K(w,s)\bar{w}(s)ds$$

for some $E: \mathcal{O} \to \mathbb{R}$ and $K: \mathcal{O} \times [0,\infty) \to \mathbb{R}$ such that $\int_0^\infty K(w,s)^2 h(s)^{-1} ds < \infty$ for each $w \in \mathcal{O}$. We assume that E and K are twice continuously differentiable on $\mathcal{O}$ and $\mathcal{O}\times[0,\infty)$, respectively, and define

$$(5.4) \qquad G(0,s) = E(0) - \int_0^s K(0,\xi)d\xi,$$

and

$$(5.5) \qquad G_\infty(0) = E(0) - \int_0^\infty K(0,\xi)d\xi.$$

Physically natural assumptions are

$$(5.6) \qquad E(0) > 0, \quad G_\infty(0) > 0,$$

$$(5.7) \qquad (-1)^k \frac{d^k}{ds^k} G(0,s) \geq 0, \quad s \geq 0, \quad k=0,1,2$$

and the history dependence will be "dissipative" if

$$(5.8) \qquad \frac{d}{ds} G(0,s)\Big|_{s=0} < 0.$$

Roughly speaking, (5.6),(5.7), and (5.8) say that the linearization of (5.1) about the zero history is the constitutive relation for a physically reasonable linear viscoelastic material of the Boltzmann type.

In order to establish existence of solutions to the corresponding equation of motion, we require that E and K satisfy certain technical conditions. In particular, we assume that there is a ball $\mathcal{O}_1$ of radius r_1 centered at zero in V_h and a locally bounded function $P:[0,r_1) \to \mathbb{R}$ such that*

(5.9) $$|E(w)| \leq P(\|w\|_h),$$

(5.10) $$|E'(w;z_1)| \leq P(\|w\|_h)\cdot\|z_1\|_h,$$

(5.11) $$|E''(w;z_1;z_2)| \leq P(\|w\|_h)\cdot\|z_1\|_h\cdot\|z_2\|_h,$$

(5.12) $$\int_0^\infty K(w,s)^2 h(s)^{-1} ds \leq P(\|w\|_h)$$

(5.13) $$\int_0^\infty \dot{K}(w,s)^2 h(s)^{-1} ds \leq P(\|w\|_h)$$

(5.14) $$\int_0^\infty K'(w;z_1,s)^2 h(s)^{-1} ds \leq P(\|w\|_h)\cdot\|z_1\|_h^2$$

(5.15) $$\int_0^\infty \dot{K}'(w;z_1,s)^2 h(s)^{-1} ds \leq P(\|w\|_h)\cdot\|z_1\|_h^2$$

(5.16) $$\int_0^\infty K''(w;z_1;z_2,s)^2 h(s)^{-1} ds \leq P(\|w\|_h)\cdot\|z_1\|_h^2\cdot\|z_2\|_h^2$$

$$\forall\ w \in \mathcal{O}_1, z_1, z_2 \in V_h,$$

and that

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* We use $K'(\cdot;\cdot,s)$ to denote the Fréchet derivative of $K(\cdot,s)$ holding s fixed and $\dot{K}(w,\cdot)$ to denote the derivative of $K(w,\cdot)$ holding w fixed. We use $\dot{K}'$ to denote the "mixed derivative".

$$(5.17) \qquad \int_0^\infty \{G(0,s)-G_\infty(0)\}^2 h(s)^{-1} ds < \infty .$$

We feel that these conditions are quite reasonable.

Many of the functions in this chapter are introduced originally as mappings from [0,1] cross a time interval into $\mathbb{R}$. Such functions can also be regarded in a natural way as mappings from a time interval into various spaces of functions defined on [0,1]. We use the same symbol to denote each of these maps. Throughout this chapter, $H^k(0,1)$ stands for the usual Sobolev space $W^{k,2}(0,1)$.

Consider now the history-boundary value problem of place, viz.

$$(5.18) \qquad u_{tt}(x,t) = \frac{\partial}{\partial x} \mathscr{G}(u_x^t(x,\cdot))+f(x,t), \; 0 < x < 1, \; t > 0,$$

$$(5.19) \qquad u(x,t) = v(x,t), \quad 0 \leq x \leq 1, \; t \leq 0,$$

$$(5.20) \qquad u(0,t) = u(1,t) = 0, \quad -\infty < t < \infty ,$$

where f is the (known) body force and v is an assigned function on $[0,1]\times(-\infty,0]$. Of f we assume that

$$(5.21) \qquad f \in C^0([0,\infty);H^1(0,1)) \cap L^2([0,\infty);H^1(0,1)),$$

$$(5.22) \qquad f_t \in C^0([0,\infty);L^2(0,1)) \cap L^2([0,\infty);L^2(0,1)),$$

$$(5.23) \qquad f_{tt} \in L^2([0,\infty);L^2(0,1)),$$

$$(5.24) \qquad f(0,t) = f(1,t) = 0, \quad t \geq 0,$$

and of v we assume that

$$(5.25)\qquad v \in \bigcap_{k=0}^{3} C^{3-k}((-\infty,0];H^k(0,1)),$$

$$(5.26)\qquad v(0,t) = v_{xx}(0,t) = v(1,t) = v_{xx}(1,t) = 0, \quad t \leq 0,$$

$$(5.27)\qquad v_{tt}(x,0) = \frac{\partial}{\partial x}\mathcal{G}(v_x(x,\ ))+f(x,0), \quad 0 \leq x \leq 1.$$

We measure the sizes of f and v by

$$(5.28)\qquad \mathcal{F}(f) = \sup_{t\in[0,\infty)} \int_0^1 (f_x^2+f_t^2)(x,t)dx + \int_0^\infty\int_0^1 (f_x^2+f_t^2+f_{tt}^2)(x,t)dxdt,$$

and

$$(5.29)\qquad \mathcal{V}(v) = \int_0^1 (v_{xxx}^2+v_{xxt}^2+v_{xtt}^2)(x,0)dx + \int_0^\infty\int_0^1 h(t)\{v_{xxx}^2+v_{xxt}^2+v_{xtt}^2\}(x,-t)dxdt.$$

Theorem 5.1: Assume that the maps $E:\mathcal{O}\to\mathbb{R}$, $K:\mathcal{O}\times[0,\infty)\to\mathbb{R}$ are twice continuously differentiable, that (5.6) through (5.17) hold and that the influence function is of the form

$$(5.30)\qquad h(s) = Me^{-cs}, \quad M,c > 0.$$

Then, there exists a positive constant δ such that for any f and v which satisfy (5.21) through (5.27) with

$$(5.31)\qquad \mathcal{F}(f) + \mathcal{V}(v) \leq \delta,$$

the history-boundary value problem (5.18),(5.19),(5.20) has a unique solution $u\in C^2([0,1]\times(-\infty,\infty))$. Moreover, u,u_x,u_t,u_{xx},u_{xt}, and u_{tt} converge to zero uniformly on [0,1] as $t \to \infty$.

Remark 5.1: Theorem 5.1 remains valid if we drop assumption (5.30) and replace (5.31) with the strengthened smallness condition

$$(5.32) \qquad \mathcal{F}(f) + \mathcal{V}(v) + \int_0^\infty \int_0^\infty \int_0^1 h(t+s)\{v_{xxx}^2 + v_{xxt}^2 + v_{xtt}^2\}(x,-t)\,dx\,dt\,ds \le \delta.$$

Remark 5.2: Theorem 5.1 remains valid if we replace (5.7) and (5.8) with the assumption that the function m defined by $m(s) = G(0,s)-G_\infty(0)$ is a strongly positive definite kernel on $[0,\infty)$.

Proof of Theorem 5.1: Our goal is to put (5.18),(5.19),(5.20) in the abstract setting of Chapter 2 and then apply Theorem 4.1. Define

$$(5.33) \qquad \mathcal{X}_0 = L^2(0,1)$$

$$(5.34) \qquad \mathcal{X}_1 = H_0^1(0,1)$$

$$(5.35) \qquad \mathcal{X}_2 = H^2(0,1) \cap H_0^1(0,1)$$

$$(5.36) \qquad \mathcal{X}_3 = \{w \in H^3(0,1) : w(0) = w_{xx}(0) = w(1) = w_{xx}(1) = 0\}$$

equipped with the norms* given by

$$(5.37) \qquad \|w\|_k^2 = \int_0^1 \left(\frac{d^k}{dx^k} w(x)\right)^2 dx.$$

* These norms are associated in an obvious way with inner products.

The natural imbeddings $\mathscr{X}_{k+1} \subset \mathscr{X}_k$ are continuous and dense, and a simple integration by parts shows that (2.8) holds. The corresponding space $\mathscr{X}_{-1}$ is $H^{-1}(0,1)$. Let $\mathscr{L}$ and $\mathscr{V}_k$, $k=1,2,3$, be constructed from the $\mathscr{X}_k$ as in Chapter 2.

Observe that if $w \in \mathscr{V}_2$, then for each fixed $x \in [0,1]$, we have $w_x(x,\cdot) \in V_h$ and

$$(5.38) \qquad \sup_{x \in [0,1]} \| w_x(x,\cdot)\|_h \leq c_1 ||| w |||_2,$$

where c_1 is a positive constant which is independent of w. Thus if H is a continuous functional on V_h, then for $w \in \mathscr{V}_2$, $H(w_x(x,\cdot))$ is well-defined for each $x \in [0,1]$, and is in fact a continuous function of x.

If u is smooth, then (5.18) is equivalent to

$$(5.39) \qquad u_{tt} - E(u^t_x)u_{xx} + \int_0^\infty K(u^t_x,s)u^t_{xx}(x,s)ds = f,$$

which is of the form (2.15). However we cannot apply Theorem 4.1 directly because E is not defined on all of V_h. This purely technical inconvenience will be overcome by constructing a smooth map Ψ: $\mathscr{V}_2 \to \mathscr{V}_2$ which contracts $\mathscr{V}_2$ to a small ball and is equal to the identity on a smaller ball. We then consider

$$(5.40) \qquad u_{tt} - E(\Psi(u^t)_x)u_{xx} + \int_0^\infty K(\Psi(u^t)_x,s)u^t_{xx}(x,s)ds = f$$

in place of (5.39), and apply Theorem 4.1. We show that the history-value problem associated with (5.40) has a unique solution u which is sufficiently "small" so that $\Psi(u^t) = u^t$

for all $t \geq 0$, whence u is also a solution of (5.39).

Let $\mathcal{O}_2$ be a ball of radius r_2 centered at zero in $\mathscr{V}_2$. For $r_2 < r_1/c_1$, where c_1 is the constant in (5.38), we can define $\tilde{A}: \mathcal{O}_2 \to \mathscr{L}$ by

$$(5.41) \qquad \tilde{A}(w)z = -E(w_x)z_{xx}, \quad w \in \mathcal{O}_2,\ z \in \mathscr{X}_1.$$

Now clearly we have

$$\langle \tilde{A}(0)z, z\rangle = E(0)\|z\|_1^2 \quad \forall\ z \in \mathscr{X}_1.$$

Moreover, $\tilde{A}$ is continuously differentiable on $\mathcal{O}_2$ and $\tilde{A}'(w)$ is bounded for $w \in \mathcal{O}_2$. Consequently, there is some smaller ball $\mathcal{O}_3$, of radius $r_3 < \min(1, r_2)$ centered at zero in $\mathscr{V}_2$, and there are positive constants λ_1 and κ such that

$$(5.42) \qquad \langle \tilde{A}(w)z, z\rangle \geq \lambda_1\|z\|_1^2 \quad \forall\ w \in \mathcal{O}_3,\ z \in \mathscr{X}_1,$$

and

$$(5.43) \qquad E(\Psi(w)_x) \geq \kappa \quad \forall\ w \in \mathcal{O}_3.$$

Also, for each $w \in \mathcal{O}_3$, $\tilde{A}(w)$ is invertible with $(\tilde{A}(w))^{-1} \in \bigcap_{k=2}^{3} \mathscr{L}(\mathscr{X}_{k-2}; \mathscr{X}_k)$ and there is a constant μ_1 such that

$$(5.44) \qquad \|(\tilde{A}(w))^{-1}z\|_k \leq \mu_1\|z\|_{k-2} \quad \forall\ w \in \mathcal{O}_3,\ z \in \mathscr{X}_1.$$

Let $\phi: [0,\infty) \to [0,\infty)$ be a C^∞ smooth function which satisfies

$$(5.45) \qquad \phi(\xi) = 1, \quad 0 \leq \xi \leq \frac{r_3^2}{2},$$

$$(5.46)\qquad \phi(\xi) \geq \sqrt{\frac{\xi}{r_3}}\,, \qquad \frac{r_3^2}{2} \leq \xi \leq r_3^2\,,$$

and

$$(5.47)\qquad \phi(\xi) = \sqrt{\frac{\xi}{r_3}}\,, \qquad \xi \geq r_3^2\,,$$

and define $\Psi\colon \mathscr{V}_2 \to \mathscr{V}_2$ by

$$(5.48)\qquad \Psi(w) = \frac{w}{\phi(|||w|||_2^2)}\,.$$

Observe that Ψ is C^∞ smooth,

$$(5.49)\qquad \Psi(w) = w \quad \forall\, w \in \mathscr{V}_2 \quad \text{with} \quad |||w|||_2 \leq \frac{r_3}{2}\,,$$

and

$$(5.50)\qquad |||\Psi(w)|||_2 \leq r_3 \qquad \forall\, w \in \mathscr{V}_2.$$

Now, define $A\colon \mathscr{V}_2 \to \mathscr{L}$ and $B\colon \mathscr{V}_2 \times [0,\infty) \to \mathscr{L}$ by

$$(5.51)\qquad A(w)z = -E(\Psi(w)_x)z_{xx}, \qquad w \in \mathscr{V}_2,\ z \in \mathscr{X}_1$$

$$(5.52)\qquad B(w,s)z = K(\Psi(w)_x,s)z_{xx}, \quad w \in \mathscr{V}_2, \quad s \geq 0,\ z \in \mathscr{X}_1,$$

and set

$$(5.53)\qquad C(0,s) = \int_s^\infty B(0,\xi)d\xi\,, \qquad s \geq 0,$$

$$(5.54)\qquad F(0) = A(0) - C(0,0).$$

Note that

$$(5.55)\qquad C(0,s)z = [G(0,s)-G_\infty(0)]z_{xx}, \quad s \geq 0,\ z \in \mathscr{X}_1,$$

and

(5.56) $$F(0)z = G_\infty(0)z_{xx}\,, \quad z \in \mathscr{X}_1.$$

It follows immediately from our construction of A,B,C, and F, and (5.6) through (5.17) that (a-1),(a-2),(a-3),(a-7),(a-8), and (a-9) are satisfied, and a simple computation shows that (a-4) is satisfied. As regards (a-5), the linear initial value problem (2.26),(2.27) here takes the form

(5.57) $$Z_{tt}(x,t) - E(\Psi(w^t)_x)Z_{xx}(x,t) = g(x,t),$$

$$0 \le x \le 1,\ 0 \le t \le T,$$

(5.58) $$Z(0,t) = Z(1,t) = 0, \quad 0 \le t \le T,$$

(5.59) $$Z(x,0) = Z_0(x), \quad Z_t(x,0) = Z_1(x), \quad 0 \le x \le 1.$$

If $w \in \bigcap_{k=0}^{3} W^{3-k,\infty}((-\infty,T];\mathscr{X}_k)$, then the function $a:[0,1]\times[0,T] \to \mathbb{R}$ defined by

(5.60) $$a(x,t) = E(\Psi(w^t)_x), \quad 0 \le x \le 1,\ 0 \le t \le T$$

satisfies

(5.61) $$a \in W^{1,\infty}([0,1]\times[0,T]),$$

and

(5.62) $$a(x,t) \ge \kappa > 0, \quad 0 \le x \le 1,\ 0 \le t \le T,$$

By standard theory for linear hyperbolic equations, (5.57), (5.58),(5.59) has a unique solution $Z \in \bigcap_{k=0}^{3} C^{3-k}([0,T];\mathscr{X}_k)$, provided that $Z_0 \in \mathscr{X}_3$, $Z_1 \in \mathscr{X}_2$, and g satisfies (2.24),(2.25). Thus (a-5) is satisfied.

It remains only to check (a-10) and (a-11). Define $m:[0,\infty) \to \mathbb{R}$ by

$$m(s) = G(0,s)-G_\infty(0), \quad s \geq 0. \tag{5.63}$$

Then, by (5.7),(5.8),(5.12),(5.13), and (5.17), m satisfies

$$(-1)^k \frac{d^k}{ds^k} m(s) \geq 0, \quad s \geq 0, \quad k=0,1,2 \tag{5.64}$$

$$\frac{d}{ds} m(s) \not\equiv 0, \quad s \geq 0, \tag{5.65}$$

and

$$m \in W^{2,1}(0,\infty), \tag{5.66}$$

From Corollary 2.2 of [14], we deduce that m is a strongly positive definite kernel on $[0,\infty)$. It now follows from Lemma 4.2 of [15] that (a-10) is satisfied. Moreover, Lemma 3.2 of [8] implies that the scalar Volterra operator L defined by

$$(L\chi)(t) = E(0)\chi(t) + \int_0^t m(t-\tau)\chi(\tau)d\tau \tag{5.67}$$

has a resolvent kernel which belongs to $L^1(0,\infty)$. This guarantees that (a-11) is satisfied.

It follows from (5.21) through (5.27) that f and v satisfy (4.1) through (4.5). Thus, Theorem 4.1 implies that for δ sufficiently small, the history value problem (2.15),(2.16) has a unique solution

$$(5.68)\qquad u \in \bigcap_{k=1}^{3} C^{3-k}((-\infty,\infty);\mathscr{X}_k).$$

In addition, the restriction of u to $[0,\infty)$ satisfies

$$(5.69)\qquad u \in \bigcap_{k=0}^{3} C^{3-k}([0,\infty);\ \mathscr{X}_k),$$

$$(5.70)\qquad u \in \bigcap_{k=0}^{3} W^{3-k,2}([0,\infty);\mathscr{X}_k).$$

The estimate (4.12) shows that by further restricting the size of δ if necessary, we have $|||u^t|||_2 \leq \frac{r_3}{2}$ for all $t \geq 0$ so that $\Psi(u^t) = u^t$ for all $t \geq 0$, whence u is also a solution of (5.18). Clearly u satisfies (5.19) and (5.20).

By (5.69) we have

$$(5.71)\qquad u_{xx},u_{xt},u_{tt} \in C^0((-\infty,\infty);H^1(0,1)),$$

and since the injection of $H^1(0,1)$ into $C[0,1]$ is continuous, this implies that

$$(5.72)\qquad u \in C^2([0,1]\times(-\infty,\infty)).$$

Finally, it follows from (5.68),(5.69) that as $t \to \infty$,

$$(5.73)\qquad u,u_x,u_t,u_{xx},u_{xt},u_{tt} \xrightarrow[{[0,1]}]{\text{unif.}} 0.$$

The proof is complete. ■

We now discuss boundary conditions of traction. Since the reference configuration is a natural state, we require that

(5.74) $\mathscr{G}(0) = 0.$

Consider the boundary condition

(5.75) $\sigma(x_0,t) = 0 , \quad -\infty < t < \infty ,$

$x_0 = 0$ or $x_0 = 1$. Implicit in (5.75) is the assumption that $|\varepsilon(x_0,t)|$ is small enough so that $\varepsilon^t(x_0,\cdot) \in \mathscr{O}$ for all t. Using the constitutive relation (5.1), we rewrite (5.75) as

(5.76) $\mathscr{G}(\varepsilon^t(x_0,\cdot)) = 0 , \quad -\infty < t < \infty ,$

which is a functional equation for ε. Clearly (5.76) holds if

(5.77) $\varepsilon(x_0,t) = 0, \quad -\infty < t < \infty.$

It is straightforward to verify that if $\mathscr{G}$ satisfies the assumptions of Theorem 5.1 and $\varepsilon(x_0,t) = 0$ for all $t \leq 0$, then (5.75) and (5.77) are equivalent. We assume that $\varepsilon(x_0,t) = 0$ for all $t \leq 0$ and replace (5.75) with (5.77).

Consider the history-boundary value problem

(5.78) $u_{tt}(x,t) = \frac{\partial}{\partial x} \mathscr{G}(u_x^t(x,\cdot)) + f(x,t), \quad 0 < x < 1,\ t > 0,$

(5.79) $u(x,t) = v(x,t), \quad 0 \leq x \leq 1,\ t \leq 0,$

* Conditions (5.75) and (5.77) are actually equivalent under the weaker assumption that $\varepsilon(x_0,\cdot) \in W^{1,2}(-\infty,0)$.

$$(5.80) \qquad u_x(0,t) = u_x(1,t) = 0, \qquad -\infty < t < \infty.$$

In place of (5.24) and (5.27) we assume that f and v satisfy

$$(5.81) \qquad \int_0^1 f(x,t)dx = 0, \qquad t \geq 0,$$

$$(5.82) \qquad \int_0^1 v(x;t)dx = 0, \quad t \leq 0,$$

and

$$(5.83) \qquad v_x(0,t) = v_x(1,t) = 0, \qquad t \leq 0.$$

<u>Theorem 5.2</u>: Assume that the maps $E: \mathcal{O} \to \mathbb{R}$, $K: \mathcal{O} \times [0,\infty) \to \mathbb{R}$ are twice continuously differentiable, that (5.6) through (5.?) hold and that the influence function h is of the form (5.30). Then, there is a positive constant δ_1 such that for any f and v which satisfy (5.21),(5.22),(5.25),(5.27),(5.81),(5.82), and (5.83) with

$$(5.84) \qquad \mathcal{F}(f) + \mathcal{V}(v) \leq \delta_1,$$

the history-boundary value problem (5.81),(5.82),(5.83) has a unique solution $u \in C^2([0,1]\times(-\infty,\infty))$. Moreover, $u, u_x, u_t, u_{xx}, u_{xt}$, and u_{tt} converge to zero uniformly on [0,1] as $t \to \infty$.

A similar result holds for the mixed problem

$$(5.85) \qquad u_{tt}(x,t) = \frac{\partial}{\partial x} \mathcal{G}(u_x^t(x,\cdot)) + f(x,t) \quad 0 < x < 1,\ t > 0,$$

$$(5.86) \qquad u(x,t) = v(x,t), \quad 0 \leq x \leq 1, \quad t \leq 0,$$

$$(5.87) \qquad u(0,t) = u_x(1,t) = 0, \quad -\infty < t < \infty .$$

Now, in place of (5.24) and (5.25), we assume that f and v satisfy

$$(5.88) \qquad f(0,t) = 0, \quad t \geq 0$$

and

$$(5.89) \qquad v(0,t) = v_{xx}(0,t) = v(1,t) = 0, \quad t \leq 0.$$

Theorem 5.3: Assume that the maps $E: \mathcal{O} \to \mathbb{R}$, $K: \mathcal{O} \times [0,\infty) \to \mathbb{R}$ are twice continuously differentiable, that (5.6) through (5.17) hold and that the influence function h is of the form (5.30). Then, there is a positive constant δ_2 such that for any f and v which satisfy (5.21),(5.22),(5.23),(5.25),(5.27),(5.88) and (5.89) with

$$(5.90) \qquad \mathcal{F}(f) + \mathcal{V}(v) \leq \delta_2,$$

the history-boundary value problem (5.85),(5.86),(5.87) has a unique solution $u \in C^2([0,1]\times(-\infty,\infty))$. Moreover, $u, u_x, u_t, u_{xx}, u_{xt}$, and u_{tt} converge to zero uniformly on [0,1] as $t \to \infty$.

Remarks 5.1 and 5.2 also apply to Theorems 5.2 and 5.3. The proofs of these theorems are almost identical to the proof of Theorem 5.1. Other types of boundary conditions can be handled similarly.

With certain modifications, the procedure presented here can also be used to establish global existence of smooth solutions to certain appropriate history value problems associated with the motion of multidimensional bodies composed of materials with fading memory. For n-dimensional bodies, we require spaces $\mathscr{X}_0, \mathscr{X}_1, \ldots \mathscr{X}_m$, and $\mathscr{V}_1, \mathscr{V}_2, \ldots \mathscr{V}_m$ where $m = [n/2]$, and that A and b be defined on $\mathscr{V}_{m-1}$ and $\mathscr{V}_{m-1} \times [0,\infty)$ respectively. We then seek a solution u of (2.15),(2.16) which satisfies $u \in \bigcap_{k=0}^{m} C^{m-k}((-\infty,\infty);\mathscr{X}_k)$. The required a priori estimates become extremely lengthy.

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References

[1] Coleman, B.D. and M.E. Gurtin, Waves in materials with memory. II. On the growth and decay of one-dimensional acceleration waves, Arch. Rational Mech. Anal. 19, 239-265 (1965).

[2] Coleman, B.D., M.E. Gurtin and I.R. Herrera, Waves in materials with memory. I. The velocity of one-dimensional shock and acceleration waves, Arch. Rational Mech. Anal. 19, 1-19 (1965).

[3] Coleman, B.D. and V.J. Mizel, Norms and semigroups in the theory of fading memory, Arch. Rational Mech. Anal. 23, 87-123 (1967).

[4] Coleman, B.D. and W. Noll, An approximation theorem for functionals with applications in continuum mechanics, Arch. Rational Mech. Anal. 6, 355-370 (1960).

[5] Coleman, B.D. and W. Noll, Foundations of linear visco-elasticity, Reviews Mod. Phys. 33, 239-249 (1961).

[6] Dafermos, C.M., The mixed initial-boundary value problem for the equations of nonlinear one-dimensional visco-elasticity, J. Differential Equations 6, 71-86 (1969).

[7] Dafermos, C.M. and J.A. Nohel, Energy methods for nonlinear hyperbolic Volterra integrodifferential equations, Comm. PDE 4, 219-278 (1979).

[8] Dafermos, C.M. and J.A. Nohel, A nonlinear hyperbolic Volterra equation in viscoelasticity, Am. J. Math Supplement, 87-116 (1981).

[9] Greenberg, J.M., R.C. MacCamy and V.J. Mizel, On the existence, uniqueness and stability of solutions of the equation $\sigma'(u_x)u_{xx}+\lambda u_{xtx} = \rho_0 u_{tt}$, J. Math. Mech. 17, 707-728 (1968).

[10] Lax, P.D., Development of singularities of solutions of nonlinear hyperbolic partial differential equations, J. Math. Phys. 5, 611-613 (1964).

[11] MacCamy, R.C., A model for one-dimensional, nonlinear viscoelasticity, Q. Appl. Math. 35, 21-33 (1977).

[12] MacCamy, R.C. and V.J. Mizel, Existence and nonexistence in the large of solutions of quasilinear wave equations, Arch. Rational Mech. Anal. 25, 299-320 (1967).

[13] Matsumura, A., Global existence and asymptotics of the solutions of the second order quasilinear hyperbolic equations with first order dissipation, Publ. Res. Inst. Math. Sci. Kyoto Univ., Ser. A 13, 349-379 (1977).

[14] Nohel, J.A. and D.F. Shea, Frequency domain methods for Volterra equations, Advances in Math. 22, 278-304 (1976).

[15] Staffans, O., On a nonlinear hyperbolic Volterra equation, SIAM J. Math. Anal. 11, 793-812 (1980).